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A CLASS OF MEAN VALUE FUNCTIONS*

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1. Introduction. The two problems discussed in the note by G. Pólya in this issue of the MONTHLY suggest the following two problems, of which the first at least is well known, concerning a given finite set of positive values $(a) \equiv (a_1, a_2, \cdots, a_n)$.

PROBLEM I. *Find the value x which minimizes the sum of the squares of the errors,* $\sum_{j=1}^{n}(a_j - x)^2$.

PROBLEM II. *Find the value y which minimizes the sum of the squares of the relative errors,* $\sum_{j=1}^{n}[(a_j - y)/y]^2$.

Solutions are readily obtained by considering first and second derivatives. The solutions are $x = \sum_{j=1}^{n} a_j/n$, which is the arithmetic mean of (a), and

$$y = \sum_{j=1}^{n} a_j^2 \Big/ \sum_{k=1}^{n} a_k, \tag{1}$$

which is a weighted arithmetic mean of (a) with each a_j as its own weight.

We shall investigate (1) as an instance of a continuum of mean value functions, and shall compare these means with the standard means $\mathfrak{M}_t(a)$.

2. The mean of order t. For a set of positive values $(a) \equiv (a_1, a_2, \cdots, a_n)$, the mean of order t, $-\infty \leqq t \leqq +\infty$, is defined [**4**, p. 54; **3**, p. 12] by

$$\mathfrak{M}_{-\infty}(a) = \min\,(a), \qquad \mathfrak{M}_{+\infty}(a) = \max\,(a), \qquad \mathfrak{M}_0(a) = \Big(\prod_{j=1}^{n} a_j\Big)^{1/n};$$

$$\mathfrak{M}_t(a) = \Big(\sum_{j=1}^{n} a_j^t/n\Big)^{1/t}, \text{ otherwise.}$$

The harmonic, geometric, and arithmetic means are the special cases $t = -1, 0, +1$, respectively.

We have $\mathfrak{M}_{-t}(a) = 1/\mathfrak{M}_t(1/a)$, and, for positive constants k, $\mathfrak{M}_t(ka) = k\mathfrak{M}_t(a)$.

As is well known, $\mathfrak{M}_t(a)$ is a continuous strictly increasing function of t, $-\infty \leqq t \leqq +\infty$, unless all the a_j are equal; $\mathfrak{M}_t(a)$ is a continuous increasing function of a_k, $k = 1, 2, \cdots, n$; and $\mathfrak{M}_t(a)$ satisfies Minkowski's inequality,

$$\mathfrak{M}_t(a + b) \leqq \mathfrak{M}_t(a) + \mathfrak{M}_t(b), \qquad t \geqq 1, \tag{2}$$

the sign of equality holding for $+\infty > t > 1$ if and only if (a) and (b) are proportional; the inequality sign in (2)is reversed for $t \leqq 1$.

3. Definition. For a set of positive values $(a) \equiv (a_1, a_2, \cdots, a_n)$, we define

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$\mathfrak{N}_t(a)$, $-\infty \leqq t \leqq +\infty$, by

$$\mathfrak{N}_{-\infty}(a) = \min\,(a), \qquad \mathfrak{N}_{+\infty}(a) = \max\,(a);$$

$$\mathfrak{N}_t(a) = \sum_{j=1}^{n} a_j^t \Big/ \sum_{k=1}^{n} a_k^{t-1}, \text{ otherwise.}$$

Then $\mathfrak{N}_t(a)$ is a weighted arithmetic mean of (a), with a_j^{t-1} as weight for a_j, $j=1, 2, \cdots, n$.

The function (1) is the special case $t=2$. The harmonic and arithmetic means are the cases $t=0$, 1, respectively. For $n=2$, the geometric mean is the case $t=1/2$.

We have $\mathfrak{N}_{-t}(a)=1/\mathfrak{N}_{t+1}(1/a)$, and, for positive constants k, $\mathfrak{N}_t(ka)=k\mathfrak{N}_t(a)$.

In the theory of asymptotic values it is shown [4, p. 78; 3, p. 62] that

$$\lim_{t\to-\infty} \mathfrak{N}_t(a) = \mathfrak{N}_{-\infty}(a), \qquad \lim_{t\to+\infty} \mathfrak{N}_t(a) = \mathfrak{N}_{+\infty}(a),$$

so that $\mathfrak{N}_t(a)$ is a continuous function of t, $-\infty \leqq t \leqq +\infty$; and clearly $\mathfrak{N}_t(a)$ is a continuous function of the a_k.

4. Comparison of $\mathfrak{N}_t(a)$ and $\mathfrak{M}_t(a)$. For $-\infty < t < +\infty$ we have

$$\mathfrak{N}_t(a) = \frac{[\mathfrak{M}_t(a)]^t}{[\mathfrak{M}_{t-1}(a)]^{t-1}} = \left[\frac{\mathfrak{M}_t(a)}{\mathfrak{M}_{t-1}(a)}\right]^{t-1} \mathfrak{M}_t(a),$$

whence it follows that $\mathfrak{N}_t(a) \geqq \mathfrak{M}_t(a)$ *for* $1 < t < +\infty$, *and* $\mathfrak{N}_t(a) \leqq \mathfrak{M}_t(a)$ *for* $-\infty < t < 1$, *the signs of equality holding if and only if all the* a_j *are equal. At* $t = -\infty, 1, +\infty$, *we have* $\mathfrak{N}_t(a) = \mathfrak{M}_t(a)$.

For negative values $-t$ *we have the stronger result that*

$$\mathfrak{N}_{-t}(a) = 1/\mathfrak{N}_{t+1}(1/a) \leqq 1/\mathfrak{M}_{t+1}(1/a) = \mathfrak{M}_{-t-1}(a).$$

5. $\mathfrak{N}_t(a)$ as function of t. By means of Cauchy's inequality [4, p. 54; 3, p. 16] it has been shown [3, p. 243] that for positive integers p we have $\mathfrak{N}_p(a) \leqq \mathfrak{N}_{p+1}(a)$, so that $\mathfrak{N}_p(a)$ is a non-decreasing function of the integer p.[5]We now extend this result as follows.

THEOREM 1. *The function* $\mathfrak{N}_t(a)$ *is a non-decreasing function of* t, $-\infty \leqq t \leqq +\infty$, *and is strictly increasing unless all the* a_j *are equal.*

Proof. We have

$$\left(\sum_{j=1}^{n} a_j^{t-1}\right)^2 d\mathfrak{N}_t(a)/dt = \sum_{j=1}^{n} a_j^{t-1} \sum_{k=1}^{n} a_k^t \log a_k - \sum_{j=1}^{n} a_j^t \sum_{k=1}^{n} a_k^{t-1} \log a_k$$

$$= \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} a_j^{t-1} a_k^{t-1} (a_j - a_k)(\log a_j - \log a_k),$$

which is positive unless each term vanishes.

An alternative proof, suggested by G. Pólya, is as follows. By Cauchy's inequality we have $d^2 \log \left(\sum_{j=1}^n a_j^t \right)/dt^2 \geqq 0$, the sign of equality holding if and only if all the a_j are equal. Hence $\log \left(\sum_{j=1}^n a_j^t \right)$ is a convex function of t and therefore, for $t_2 > t_1$, we have

$$\log \left(\sum_{j=1}^n a_j^{t_2} \right) - \log \left(\sum_{j=1}^n a_j^{t_2-1} \right) \geqq \log \left(\sum_{j=1}^n a_j^{t_1} \right) - \log \left(\sum_{j=1}^n a_j^{t_1-1} \right).$$

6. $\mathfrak{M}_t(a)$ as function of the a_k. We have

$$\left(\sum_{j=1}^n a_j^{t-1} \right)^2 \partial \mathfrak{M}_t(a)/\partial a_k = a_k^{t-2} \sum_{j=1}^n a_j^{t-1} [t a_k - (t-1)\mathfrak{M}_t(a)]. \tag{3}$$

Hence *for* $0 \leqq t \leqq 1$, $\mathfrak{M}_t(a)$ *is a strictly increasing function of* a_k. Further, while precise conditions for each t can be obtained from (3), $\mathfrak{M}_t(a)$ is an increasing function of a_k for $t > 1$ provided $a_k > \mathfrak{M}_t(a)$, and for $t < 0$ provided $a_k < \mathfrak{M}_t(a)$.

7. Minkowski's inequality. We shall establish the following result.

THEOREM 2. *Let* $(a) \equiv (a_1, a_2, \cdots, a_n)$ *and* $(b) \equiv (b_1, b_2, \cdots, b_n)$ *be sets of positive values. Then*

$$\mathfrak{M}_t(a + b) \leqq \mathfrak{M}_t(a) + \mathfrak{M}_t(b), \qquad 1 \leqq t \leqq 2, \tag{4}$$

and

$$\mathfrak{M}_t(a + b) \geqq \mathfrak{M}_t(a) + \mathfrak{M}_t(b), \qquad 0 \leqq t \leqq 1, \tag{5}$$

the signs of equality holding if and only if $t = 1$ *or* (a) *is proportional to* (b).

Proof. For the function

$$\phi(\lambda) \equiv \mathfrak{M}_t[\lambda a + (1 - \lambda)b], \qquad 0 \leqq \lambda \leqq 1,$$

we have

$$\begin{aligned} \left(\sum_{j=1}^n A_j^{t-1} \right)^3 d^2\phi/d\lambda^2 = {} & (t-1)\left\{ 2\left[\sum_{j=1}^n A_j^t \sum_{k=1}^n A_k^{t-1} \sum_{l=1}^n A_l^{t-2} B_l^2 \right.\right. \\ & \left. - \sum_{j=1}^n A_j^t \left(\sum_{k=1}^n A_k^{t-2} B_k \right)^2 \right] - t\left[\sum_{j=1}^n A_j^t \sum_{k=1}^n A_k^{t-1} \sum_{l=1}^n A_l^{t-2} B_l^2 \right. \\ & + 2 \sum_{j=1}^n A_j^{t-1} \sum_{k=1}^n A_k^{t-2} B_k \sum_{l=1}^n A_l^{t-1} B_l - \left(\sum_{j=1}^n A_j^{t-1} \right)^2 \sum_{k=1}^n A_k^{t-2} B_k^2 \\ & \left.\left. - 2 \sum_{j=1}^n A_j^t \left(\sum_{k=1}^n A_k^{t-2} B_k \right)^2 \right] \right\} \\ = {} & (t-1)(2P - tQ), \end{aligned} \tag{6}$$

where $A_j=\lambda a_j+(1-\lambda)b_j$, $B_j=a_j-b_j$, and P and Q are the indicated expressions in brackets.

By Cauchy's inequality we have $P\geqq 0$. Further, first by the inequality between the arithmetic and geometric means and then by Cauchy's inequality, we have

$$P-Q=\Big(\sum_{j=1}^{n}A_j^{t-1}\Big)^2\sum_{k=1}^{n}A_k^{t-2}B_k^2+\sum_{j=1}^{n}A_j^{t}\Big(\sum_{k=1}^{n}A_k^{t-2}B_k\Big)^2$$

$$-2\sum_{j=1}^{n}A_j^{t-1}\sum_{k=1}^{n}A_k^{t-2}B_k\sum_{l=1}^{n}A_l^{t-1}B_l$$

$$\geqq 2\sum_{j=1}^{n}A_j^{t-1}\sum_{k=1}^{n}A_k^{t-2}B_k\left\{\Big[\sum_{l=1}^{n}A_l^{t-2}B_l^2\sum_{m=1}^{n}A_m^{t}\Big]^{1/2}-\sum_{l=1}^{n}A_l^{t-1}B_l\right\}\geqq 0.$$

Thus for $0\leqq t\leqq 2$ we have $2P-tQ\geqq 0$, whence, by (6), $\phi(\lambda)$ is a convex function of λ for $1\leqq t\leqq 2$, and a concave function of λ for $0\leqq t\leqq 1$. Hence $2\phi(1/2)\leqq\phi(0)+\phi(1)$ for $1\leqq t\leqq 2$, and $2\phi(1/2)\geqq\phi(0)+\phi(1)$ for $0\leqq t\leqq 1$; then (4) and (5) follow.

The conditions for the signs of equality to hold in the inequalities used in the proof reduce to the conditions expressed in the theorem.

An alternative method of proof for the interval $1\leqq t\leqq 2$ was suggested by R. Bellman, who has kindly supplied the details [1]. A third method for this interval, suggested by G. Forsythe, is included in the next section because of its geometric aspects.

8. An alternative proof. By Minkowski's lemma [2, p. 23], if a positive vector function $\|x\|$ satisfies $\|kx\|=k\|x\|$ for all $k>0$, and if the "unit sphere" $\|x\|=1$ is convex, then this function satisfies $\|x+y\|\leqq\|x\|+\|y\|$. For under the hypotheses we have

$$\|x+y\|=[\|x\|+\|y\|]\cdot\left\|\frac{\|x\|}{\|x\|+\|y\|}\,\frac{x}{\|x\|}+\frac{\|y\|}{\|x\|+\|y\|}\,\frac{y}{\|y\|}\right\|\leqq\|x\|+\|y\|.$$

For the function $\|a\|\equiv\mathfrak{N}_t(a)$, $1\leqq t\leqq 2$, let $\|a\|=\|b\|=1$, with $a_j>0$ and $b_j>0$, $j=1, 2, \cdots, n$. Then a_j^t is a convex function, and a_j^{t-1} a concave function, of a_j, whence, as observed by W. Seidel, for $0\leqq\lambda\leqq 1$ we have the following relationship:

$$\|\lambda a+(1-\lambda)b\|=\frac{\sum_{j=1}^{n}[\lambda a_j+(1-\lambda)b_j]^t}{\sum_{j=1}^{n}[\lambda a_j+(1-\lambda)b_j]^{t-1}}\leqq\frac{\lambda\sum_{j=1}^{n}a_j^t+(1-\lambda)\sum_{j=1}^{n}b_j^t}{\lambda\sum_{j=1}^{n}a_j^{t-1}+(1-\lambda)\sum_{j=1}^{n}b_j^{t-1}}=1,$$

and (4) follows from Minkowski's lemma.

For example, with $t=2$, $n=2$, $x>0$, $y>0$, the "unit sphere" $(x^2+y^2)/(x+y)=1$ is the portion of the circle $(x-1/2)^2+(y-1/2)^2=1/2$ which lies in the first

quadrant; this is convex. But the boundary would not be convex if we included all quadrants: there are *real* sets (a) and (b) for which $\mathfrak{M}_2(|a+b|)>\mathfrak{M}_2(|a|)+\mathfrak{M}_2(|b|)$.

9. Other values of t. Clearly the inequalities (4) and (5) hold for $t=+\infty$ and $t=-\infty$, respectively. But it is easy to show that for any t satisfying $2<t<+\infty$ there is an $\epsilon>0$ such that for $(a)=(1, \epsilon, \cdots, \epsilon)$ and $(b)=(1, \epsilon^2, \cdots, \epsilon^2)$ the inequality (4) does not hold. Similarly for any t satisfying $-\infty<t<0$ there is an $M>0$ such that the inequality (5) does not hold for $(a)=(1, M, \cdots, M)$, $(b)=(1, M^2, \cdots, M^2)$. Nevertheless we have the following result supplied by W. Seidel.

THEOREM 3. *If the components of (b) are all equal, $(b)=(b, b, \cdots, b)$, then*

$$\mathfrak{M}_t(a+b) \leqq \mathfrak{M}_t(a)+\mathfrak{M}_t(b), \qquad 1<t<+\infty, \tag{7}$$

and

$$\mathfrak{M}_t(a+b) \geqq \mathfrak{M}_t(a)+\mathfrak{M}_t(b), \qquad -\infty<t<1, \tag{8}$$

the signs of equality holding if and only if all components of (a) are equal.

Proof. For the function $f(x)=\sum_{j=1}^n (a_j+x)^t / \sum_{k=1}^n (a_k+x)^{t-1}$, we have

$$\frac{df}{dx}=\frac{t\left[\sum_{j=1}^n (a_j+x)^{t-1}\right]^2-(t-1)\sum_{j=1}^n (a_j+x)^t \sum_{k=1}^n (a_k+x)^{t-2}}{\left[\sum_{j=1}^n (a_j+x)^{t-1}\right]^2}.$$

It follows from Cauchy's inequality that for $t>1$ we have $df/dx \leqq t-(t-1)=1$, whence $f(b) \leqq f(0)+b$, which is (7). The sign of equality holds if and only if the sets $(a_j+b)^{t/2}$ and $(a_j+b)^{t/2-1}$ are proportional, that is, if and only if all the a_j are equal. We obtain (8) similarly.

10. Generalizations. The above results hold also for weighted means,

$$\mathfrak{M}_t(a, p) \equiv \frac{\sum_{j=1}^n p_j a_j^t}{\sum_{k=1}^n p_k a_k^{t-1}}, \qquad p_j>0,$$

for simply or doubly infinite sequences $(a)\equiv(a_1, a_2, \cdots)$ or $(a)\equiv(\cdots a_{-1}, a_0, a_1, \cdots)$, and for integral means

$$\mathfrak{M}_t(f, p) \equiv \frac{\int_a^b p(x)[f(x)]^t dt}{\int_a^b p(x)[f(x)]^{t-1} dt}, \qquad p(x) > 0.$$

References

1. Richard Bellman, Some elementary inequalities, U. S. Airforce Project Rand, D-484, The Rand Corporation, Santa Monica ,Calif.
2. T. Bonnesen und W. Fenchel, Theorie der Konvexen Körper, Springer, Berlin, 1934.
3. G. H. Hardy, J. E. Littlewood, G. Pólya, Inequalities, Cambridge, England, 1934.
4. G. Pólya und G. Szegö, Aufgaben und Lehrsätze aus der Analysis, vol. I, Springer, Berlin, 1925.

ERRATA

The following errata in recent volumes have been called to the attention of the editors:

Edmund Pinney, *Vibration Modes of Tapered Beams*, vol. 54.

p. 393, formula (10), change sign of both μ's.

p. 393, formula (13), change sign of μ's in $\beta\nu - \alpha$ column.

p. 393, formula (15), replace $x^{\mu/2}$ by $x^{-\mu/2}$.

p. 394, formula (17), replace left member of equation by

$$xF_{\mu+1}(A, B, C, D; x) + F_{\mu-1}(A, B, -C, -D; x).$$

P. M. Bailey, abstract of the paper *A general finite difference summation formula*, vol. 55.

p. 658, replace the displayed equations by

$$\sum_{x=\alpha}^{\alpha+(n-1)r}{}_r\ \phi(x) = \sum_{z=0}^{(h/r)-1}{}_1\ \Delta_n f(\alpha + xr)$$

$$\sum{}_r\ \phi(x) = \sum_{k=0}^{(h/r)-1}{}_1\ f(x + kr) + C.$$

Victor Thebault, *Concerning Two Classes of Remarkable Perfect Square Pairs*, vol. 56.

p. 444, (ii), replace $x^2 = 2\ \overline{46}\ 4\ \overline{46}$ by $x^2 = 4\ \overline{46}\ 4\ \overline{46}$.

Problem 4224 [1946, 537], vol. 56.

p. 348. line 17, replace the summation limits $j=1$ to n by $j=0$ to $c-1$.

JAMES HUME AND EXPONENTS

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A widely-used work of reference, Cajori's *History of Mathematical Notations*, reports that "Hume took an important step in his edition of *L'algèbre de Viète* (Paris, 1636), in which he wrote A^{iii} for A^3. Except for the use of Roman numerals one has here the notation used by Descartes in 1637 in his *La géométrie*." [1] Further on Cajori makes this same assertion more explicitly: "In 1636 James Hume brought out an edition of the algebra of Vieta, in which he introduced a superior notation, writing down the base and elevating the exponent to a position above the regular line and a little to the right . . . Thus, this Scotsman, residing in Paris, had almost hit upon the exponential symbolism which has become universal through the writings of Descartes . . . Hérigone and Hume almost hit upon the scheme of Descartes. The only difference was, in one case, the position of the exponent, and, in the other, the exponent written in Roman numerals. Descartes expressed the exponent in Arabic numerals and assigned it an elevated position. Where Hume would write $5a^{iv}$ and Hérigone would write 5a4, Descartes wrote $5a^4$." [2] Years after reading these sentences, I had occasion to refer to the work of Hume in question and was struck by the absence of elevated exponents. It would appear that Cajori had been misled by a passage from the *Oeuvres de Descartes* [3] in which the use of "exposants marqués en chiffres romains" is traced to Hume's *algèbre de Viète*. Wieleitner, too, jumped to the erroneous conclusion that Hume's exponents were elevated, for he refers to the same treatise of 1636 as one in which "römische Ziffern als wirkliche Exponenten benutzt wurden, z. B. in der Gleichung A^{III}—AAH égal à X." [4] Hume did indeed use Roman numerals as indices of powers. He wrote that "Aij signifie que la grandeur Aij est quarré, ou bien le quarré de la grandeur Aj"; in the same way "Aiij signifie le cube de la grandeur Aj"; and "Ajv signifie le quarré de quarré de Aj"; and similarly for higher powers [5]. This notation is used throughout, so that the equation which now would be written as $x^4-a^3x=b^4$ appears [6] in the form Ajv−Diij en Aj est egal à Zjv. Occasional unevenness of the type results in the placing of some of the exponents (*logarymes*) slightly above or below the letter to which they apply; but there obviously is no intentional elevation as in the case of Cartesian exponents in *La géométrie*. Until new evidence is presented, it appears that Descartes, not Hume, was the first mathematician to adopt exponents in the modern sense, namely, as "a symbol written above another symbol and on the right."

References

1. Florian Cajori, *A history of mathematical notations* (2 vols., Chicago, 1928–1929), I, 204. In my paper on "Fractional indices, exponents, and powers," *National Mathematics Magazine*, XVIII (1943), 81–86, the error of Cajori unfortunately is repeated. See also V. B. Bly, "A history of the exponent," *Bulletin of the Kansas Association of Teachers of Mathematics*, XXIII (1949), 20–23.

2. Cajori, *op. cit.*, I, 345–346.

3. Ed. by Charles Adam and Paul Tannery (12 vols. and supplement, Paris, 1897–1913), V, 504–512. See especially page 504.

4. Heinrich Wieleitner, *Geschichte der Mathematik* (2 vols., Leipzig, 1908–1921), vol. II, part 1, page 4.

5. *Algèbre de Viète, d'une methode nouvelle, claire, et facile* (Paris, 1636), page 3.

6. *Ibid.*, page 550.

A THEOREM ON SETS OF COPRIME INTEGERS

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1. Introduction. Let n, s, r be integers such that $n \geqq 1$, $2 \leqq r \leqq s$, and denote by $N_{s,r}(n)$ the number of representations (order being regarded as relevant) of n as the sum of s positive integers such that the highest common factor of any r of them is equal to 1. The object of this note is to investigate the asymptotic behavior of $N_{s,r}(n)$ when s, r are fixed and $n \to \infty$.

It will be convenient to write

$$\sum \mathfrak{C} \mid f(m, n, \cdots)$$

in place of

$$\sum_{\mathfrak{C}} f(m, n, \cdots);$$

here $\mathfrak{C}$ stands for the set of conditions which define the range of summation. The highest common factor of $n_1, \cdots, n_s$ will be denoted by $(n_1, \cdots, n_s)$.

The case $r = s$ is dealt with very easily. Writing

$$N_{s,s}(n; d) = \sum \begin{matrix} n = n_1 + \cdots + n_s \\ (n_1, \cdots, n_s) = d \end{matrix} \mid 1,$$

so that, in particular, $N_{s,s}(n; 1) = N_{s,s}(n)$, we have

$$N_{s,s}(n; d) = \sum \begin{matrix} n/d = \nu_1 + \cdots + \nu_s \\ (\nu_1, \cdots, \nu_s) = 1 \end{matrix} \mid 1 = N_{s,s}\left(\frac{n}{d}\right).$$

Hence

$$\sum_{d \mid n} N_{s,s}\left(\frac{n}{d}\right) = \sum n = n_1 + \cdots + n_s \mid 1 = M_s(n),$$

where

$$M_s(n) = \frac{(n-1)(n-2)\cdots(n-s+1)}{(s-1)!} = \sum_{0\leqq i\leqq s-1} A_i n^{s-1-i}.$$

Therefore, by Möbius's inversion formula,

$$\begin{aligned} N_{s,s}(n) &= \sum_{d\mid n} \mu(d) M_s\left(\frac{n}{d}\right) = \sum_{d\mid n} \mu(d) \sum_{0\leqq i\leqq s-1} A_i \left(\frac{n}{d}\right)^{s-1-i} \\ &= \sum_{0\leqq i\leqq s-1} A_i n^{s-1-i} \sum_{d\mid n} \frac{\mu(d)}{d^{s-1-i}} \\ &= \sum_{0\leqq i\leqq s-1} A_i n^{s-1-i} \prod_{p\mid n}\left(1 - \frac{1}{p^{s-1-i}}\right) \\ &= \frac{n^{s-1}}{(s-1)!} \prod_{p\mid n}\left(1 - \frac{1}{p^{s-1}}\right) + O(n^{s-2}). \end{aligned}$$

It is also easy to see that for $s=2$, $n>1$, the error term in this formula is identically equal to zero.

The case $2\leqq r<s$ is more interesting. We shall write

$$a_k^{(s,r)} = (-1)^{r+k-1}\binom{s}{r+k} \sum_{0\leqq i\leqq r-1} (-1)^i \binom{r+k}{i} \qquad (0 \leqq k \leqq s-r),$$

$$\mathrm{X}_{s,r}(p) = a_0^{(s,r)} p^{s-r-1} + a_1^{(s,r)} p^{s-r-2} + \cdots + a_{s-r-1}^{(s,r)},$$

$$\mathfrak{S}_{s,r}(n) = \prod_{p\mid n}\left(1 - \frac{\mathrm{X}_{s,r}(p)}{p^{s-1}}\right) \prod_{p\mid n}\left(1 - \frac{\mathrm{X}_{s,r}(p) + a_{s-r}^{(s,r)}}{p^{s-1}}\right).$$

The following result will be proved.

THEOREM. *Let $2\leqq r<s$. Then, as $n\to\infty$,*

$$N_{s,r}(n) = \frac{n^{s-1}}{(s-1)!}\mathfrak{S}_{s,r}(n) + O\left(\frac{n^{s-1}}{\log^{r-1} n \log\log n}\right),$$

where the O-constant depends at most on s.

Furthermore, the true order of magnitude of $N_{s,r}(n)$ is n^{s-1}.

2. Notation. The letter x denotes a positive number; all other small letters denote integers, which are positive unless the contrary is stated. The letter p is reserved for primes. As usual $\pi(x)$ denotes the number of primes not exceeding x.

The O-notation refers to the passage $n\to\infty$ unless otherwise stated, and the O-constants depend at most on s.

We shall frequently write $a\equiv b(\cdot m)$ as an abbreviation for $a\equiv b \pmod m$.

If $a_1, \cdots, a_s$ are non-negative integers, then $[a_1, \cdots, a_s]$ denotes the number of zeros among them.

For $s\geqq 1$, $n\geqq 1$, any p, we write

$$\tau_s(n, p) = \sum_{\substack{0 < \lambda_1, \cdots, \lambda_s < p \\ \lambda_1 + \cdots + \lambda_s \equiv n(\cdot p)}} 1.$$

For $2 \leqq r < s$, $n \geqq 1$, any p, we write

$$t_{s,r}(n, p) = \sum_{\substack{0 \leqq \lambda_1, \cdots, \lambda_s < p \\ \lambda_1 + \cdots + \lambda_s \equiv n(\cdot p); \quad [\lambda_1, \cdots, \lambda_s] < r}} 1.$$

3. Lemmas.

LEMMA 1.

$$\tau_s(n, p) = \begin{cases} \dfrac{(p-1)^s - (-1)^s}{p} & \text{if } p \nmid n, \\ \dfrac{(p-1)^s + (-1)^s(p-1)}{p} & \text{if } p \mid n. \end{cases}$$

Proof.

$$\tau_1(n, p) = \sum_{\substack{0 < \lambda < p \\ \lambda \equiv n(\cdot p)}} 1 = \begin{cases} 1 & \text{if } p \nmid n, \\ 0 & \text{if } p \mid n. \end{cases}$$

Hence the lemma is true for $s = 1$. Assume next that it is true for some $s \geqq 1$. Then

$$\begin{aligned} \tau_{s+1}(n, p) &= \sum_{0<\lambda<p} \sum_{\substack{0 < \lambda_1, \cdots, \lambda_s < p \\ \lambda_1 + \cdots + \lambda_s \equiv n - \lambda(\cdot p)}} 1 \\ &= \sum_{0<\lambda<p} \tau_s(n - \lambda, p). \end{aligned}$$

If $p \nmid n$, then

$$\left.\begin{array}{l} p \nmid (n - \lambda) \text{ for } p - 2 \text{ values} \\ p \mid (n - \lambda) \text{ for one value} \end{array}\right\} \text{of } \lambda \text{ in the range } 0 < \lambda < p.$$

Hence

$$\begin{aligned} \tau_{s+1}(n, p) &= (p-2)\frac{(p-1)^s - (-1)^s}{p} + \frac{(p-1)^s + (-1)^s(p-1)}{p} \\ &= \frac{(p-1)^{s+1} - (-1)^{s+1}}{p}. \end{aligned}$$

If $p \mid n$, then $p \nmid (n-\lambda)$ for $0 < \lambda < p$, and so

$$\tau_{s+1}(n, p) = (p-1)\frac{(p-1)^s - (-1)^s}{p} = \frac{(p-1)^{s+1} + (-1)^{s+1}(p-1)}{p}.$$

This completes the proof of Lemma 1.

LEMMA 2.

$$\frac{t_{s,r}(n,\ p)}{p^{s-1}} = \begin{cases} 1 - \dfrac{X_{s,r}(p)}{p^{s-1}} & \text{if } p \nmid n, \\ 1 - \dfrac{X_{s,r}(p) + a_{s-r}^{(s,r)}}{p^{s-1}} & \text{if } p \mid n. \end{cases}$$

Proof. We have

$$t_{s,r}(n,\ p) = \sum_{0 \le i \le r-1} \sum_{\substack{0 \le \lambda_1, \cdots, \lambda_s < p; \quad [\lambda_1, \cdots, \lambda_s] = i \\ \lambda_1 + \cdots + \lambda_s \equiv n (\cdot p)}} 1$$

$$= \sum_{0 \le i \le r-1} \binom{s}{i} \tau_{s-i}(n,\ p), \tag{1}$$

and the required result now follows by a straightforward calculation on making use of Lemma 1.

LEMMA 3. *There exist positive constants C_1, C_2 (depending at most on s) such that, for $2 \le r < s$, $n \ge 1$,*

$$C_1 < \mathfrak{S}_{s,r}(n) < C_2.$$

Proof. By Lemma 1 we see that $\tau_s(n,\ p) \ge 0$ always, and $\tau_s(n,\ p) > 0$ unless $p = 2,\ 2 \mid n,\ 2 \nmid s$ or $p = 2,\ 2 \nmid n,\ 2 \mid s$. Hence, for $2 \le r < s$, $n \ge 1$, we have, using (1),

$$t_{s,r}(n,\ p) \ge \tau_s(n,\ p) + s\tau_{s-1}(n,\ p) > 0.$$

Furthermore, by Lemma 2,

$$\frac{t_{s,r}(p)}{p^{s-1}} = 1 + O\left(\frac{1}{p^r}\right) \qquad (p \to \infty), \tag{2}$$

and

$$\mathfrak{S}_{s,r}(n) = \prod_p \frac{t_{s,r}(p)}{p^{s-1}}. \tag{3}$$

The assertion therefore follows.

4. Proof of the theorem. Let x be a function of n, to be fixed later, such that $x \to \infty$ as $n \to \infty$, and

$$s \prod_{p \le x} p < n.$$

Denote by $p_1, \cdots, p_k$ the primes not exceeding x.

Let $N^{(1)} = N^{(1)}(n,\ s,\ r)$ be the number of representations of n as the sum of s positive integers, no r of which are divisible by any $p \le x$. Furthermore, let $N^{(2)} = N^{(2)}(n,\ s,\ r)$ be the number of representations of n as the sum of s positive

integers, at least r of which are divisible by some $p>x$.

We clearly have

$$0 \leqq N^{(1)} - N_{s,r}(n) \leqq N^{(2)}. \tag{4}$$

Moreover,

$$\begin{aligned} N^{(2)} &\leqq \sum_{p>z} \sum_{\substack{n_1+\cdots+n_s=n \\ \text{At least } r \text{ among } n_1,\cdots,n_s \text{ are divisible by } p}} 1 \\ &\leqq \binom{s}{r} \sum_{p>z} \sum_{\substack{n_1+\cdots+n_s=n \\ n_1\equiv\cdots\equiv n_r\equiv 0(\cdot p)}} 1 \\ &= O\Big(\sum_{p>z} \sum_{\nu_1 p+\cdots+\nu_r p+\nu_{r+1}+\cdots+\nu_s=n} 1 \Big) \\ &= O\Big(\sum_{p>z} \frac{n^r}{p^r} n^{s-1-r} \Big) = O\Big(n^{s-1} \sum_{p>z} \frac{1}{p^r} \Big). \end{aligned}$$

But, using the well-known estimate* $\pi(x) = O(x/\log x)$, we have

$$\begin{aligned} \sum_{p>z} \frac{1}{p^r} &= \sum_{n>z} \frac{\pi(n) - \pi(n-1)}{n^r} \leqq \sum_{n>z} \pi(n) \Big(\frac{1}{n^r} - \frac{1}{(n+1)^r} \Big) \\ &= \sum_{n>z} O\Big(\frac{n}{\log n} \Big) O\Big(\frac{1}{n^{r+1}} \Big) = O\Big(\frac{1}{x^{r-1} \log x} \Big). \end{aligned}$$

Therefore

$$N^{(2)} = O\Big(\frac{n^{s-1}}{x^{r-1} \log x} \Big),$$

and so, by (4),

$$N_{s,r}(n) = N^{(1)} + O\Big(\frac{n^{s-1}}{x^{r-1} \log x} \Big). \tag{5}$$

To estimate $N^{(1)}$ we observe that if $n = n_1 + \cdots + n_s$ and p is any given prime, then fewer than r among $n_1, \cdots, n_s$ are divisible by p if and only if there exist numbers $\lambda_1, \cdots, \lambda_s$ such that

$$0 \leqq \lambda_i < p, \qquad n_i \equiv \lambda_i \pmod{p} \qquad (1 \leqq i \leqq s),$$
$$[\lambda_1, \cdots, \lambda_s] < r, \qquad \lambda_1 + \cdots + \lambda_s \equiv n \pmod{p}.$$

Hence we have

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* For a proof see, for instance, G. H. Hardy and E. M. Wright, An introduction to the theory of numbers (Second edition, Oxford, 1945), §22.7.

$$N^{(1)} = \sum_{\substack{0 \leqq \lambda_{ij} < p_j;\ \lambda_{1j} + \cdots + \lambda_{sj} \equiv n (\cdot p_j) \\ [\lambda_{1j}, \cdots, \lambda_{sj}] < r;\ (1 \leqq i \leqq s;\ 1 \leqq j \leqq k)}} \Bigg| \sum_{\substack{n_1 + \cdots + n_s = n \\ n_i \equiv \lambda_{ij} (\cdot p_j)\ (1 \leqq i \leqq s;\ 1 \leqq j \leqq k)}} \Bigg| 1. \tag{6}$$

Here the inner sum J is equal to

$$J = \sum_{\substack{n_1 + \cdots + n_s = n \\ n_i \equiv \mu_i (\cdot p_1 \cdots p_k)\ (1 \leqq i \leqq s)}} \Bigg| 1,$$

where the μ's depend on the λ's. Since clearly $p_1 \cdots p_k \mid (n - \mu_1 - \cdots - \mu_s)$ and $0 \leqq \mu_1, \cdots, \mu_s < p_1 \cdots p_k$, we have

$$\begin{aligned} J &= \frac{1}{(s-1)!}\left(\frac{n - \mu_1 - \cdots - \mu_s}{p_1 \cdots p_k}\right)^{s-1} + O\left\{\left(\frac{n}{p_1 \cdots p_k}\right)^{s-2}\right\} \\ &= \frac{1}{(s-1)!}\left(\frac{n}{p_1 \cdots p_k}\right)^{s-1} + O\left\{\left(\frac{n}{p_1 \cdots p_k}\right)^{s-2}\right). \end{aligned}$$

Hence, by (6),

$$N^{(1)} = \frac{n^{s-1}}{(s-1)!} \prod_{p \leqq z} \frac{t_{s,r}(n, p)}{p^{s-1}} + O\left(n^{s-2} \prod_{p \leqq z} \frac{t_{s,r}(n, p)}{p^{s-2}}\right). \tag{7}$$

But

$$t_{s,r}(n, p) \leqq \sum 0 \leqq \lambda_1, \cdots, \lambda_s < p \Bigg| 1 \leqq p^{s-1}, \tag{8}$$

and therefore, by (7), (8), (2), and (3),

$$N^{(1)} = \frac{n^{s-1}}{(s-1)!} \mathfrak{S}_{s,r}(n) \left\{1 + O\left(\frac{1}{x^{r-1} \log x}\right)\right\} + O\left(n^{s-2} \prod_{p \leqq z} p\right). \tag{9}$$

But, as is well known,*

$$\sum_{p \leqq z} \log p \leqq 3x,$$

so that

$$\prod_{p \leqq z} p \leqq e^{3z}.$$

Hence, by (9), (5), and Lemma 3,

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* Hardy and Wright, *op. cit.*, §22.6.

$$N_{s,r}(n) = \frac{n^{s-1}}{(s-1)!}\mathfrak{S}_{s,r}(n) + O\left(\frac{n^{s-1}}{x^{r-1}\log x}\right) + O(n^{s-2}e^{3x}).$$

The choice of x is still at our disposal. It is advantageous to take $x = \frac{1}{4}\log n$; with this choice of x we obtain

$$N_{s,r}(n) = \frac{n^{s-1}}{(s-1)!}\mathfrak{S}_{s,r}(n) + O\left(\frac{n^{s-1}}{\log^{r-1} n \log\log n}\right).$$

This is our first assertion and the second now follows immediately by Lemma 3.

5. The conjugate problem. In conclusion it may be of some interest to mention the conjugate problem of the problem discussed above. This consists in the determination of an asymptotic formula for the number $F(x) = F(x; r; k_1, \cdots, k_s)$ of positive integers $n \leqq x$ which are such that the highest common factor of any r of the integers $n+k_1, \cdots, n+k_s$ is 1. An account of this problem (in a more general form) has previously appeared in this MONTHLY†.

† L. Mirsky, On coprime values taken by given polynomials, this MONTHLY, vol. 55, 1948, pp. 88–89.

A CALCULUS OF FIGURATE NUMBERS AND FINITE DIFFERENCES*

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1. Introduction. The purpose of this article is to derive an operational calculus in the field of finite differences, which is based on the properties of generalized figurate numbers. The procedures necessarily yield old as well as new results, but the emphasis will be on methods and results which are believed to be new.

2. Figurate Numbers are here generalized to include figurate numbers of negative order as well as the traditional series of positive order [1]. The more general numbers are derived by using 1, 0, 0, $\cdots$ as a generating series, and operating on it by repeated summation for figurate numbers of positive order and by repeated inverse summation for figurate numbers of negative order.

The generating series 1, 0, 0, $\cdots$ is defined as the series of figurate numbers of zero order. In our notation, superscripts will represent orders of figurate numbers, positive or negative, and subscripts 0, 1, 2, $\cdots$ will represent the successive terms. Thus, the $(r+1)$th term in the series of figurate numbers of the nth order will be represented by F_r^n. The operation of repeated summation is, of course, the process of successive cumulative addition which connects the

* Based on a presentation by the author at the Mathematics Colloquium of the University of Pittsburgh, April 24, 1947, through the courtesy of Professor J. S. Taylor, Head of the Department.

adjacent columns, or rows, of Pascal's triangle. Inverse summation, as the term implies, is the operation which reverses or "undoes" the operation of summation. Thus, regardless of the sign of the superscript, the following basic relationships are true:

$$F_r^{n+1} = F_0^n + F_1^n + F_2^n + \cdots + F_r^n \tag{1}$$

$$F_r^{n+1} = F_r^n + F_{r-1}^{n+1}, \qquad r > 0 \tag{2}$$

$$F_0^m = F_0^{-n} = 1. \tag{3}$$

Table 1 presents a short table of figurate numbers of orders ranging from -7 to 7, and terms ranging from 0 to 7.

TABLE 1

r	*Negative Order*							*Positive Order*							
	F_r^{-7}	F_r^{-6}	F_r^{-5}	F_r^{-4}	F_r^{-3}	F_r^{-2}	F_r^{-1}	F_r^0	F_r^1	F_r^2	F_r^3	F_r^4	F_r^5	F_r^6	F_r^7
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7
2	21	15	10	6	3	1	0	0	1	3	6	10	15	21	28
3	−35	−20	−10	−4	−1	0	0	0	1	4	10	20	35	56	84
4	35	15	5	1	0	0	0	0	1	5	15	35	70	126	210
5	−21	−6	−1	0	0	0	0	0	1	6	21	56	126	252	462
6	7	1	0	0	0	0	0	0	1	7	28	84	210	462	924
7	−1	0	0	0	0	0	0	0	1	8	36	120	330	792	1716

In standard combinatorial notation,

$$F_r^n = \binom{n+r-1}{r} = \binom{n+r-1}{n-1} = \binom{r+1+n-2}{n-1} = F_{n-1}^{r+1} \tag{4}$$

$$F_r^{-n} = \binom{-n+r-1}{r} = \binom{-(n-r+1)}{r} = (-1)^r \binom{n}{r} \tag{5}$$

$$= (-1)^r \binom{n}{n-r} = (-1)^n F_{n-r}^{-n}. \tag{6}$$

Conversely,

$$\binom{a}{b} = F_b^{a-b+1} = (-1)^b F_b^{-a} \tag{7}$$

and

$$\binom{-a}{b} = F_b^{-a-b+1} = F_b^{-(a+b-1)} = (-1)^b \binom{a+b-1}{b}. \tag{8}$$

In the notation of the calculus of finite differences $\Delta u_x = u_{x+1} - u_x$. Since, from (2)

$$F_{r+1}^{n-1} = F_{r+1}^{n} - F_{r}^{n}, \tag{9}$$

$$\Delta F_{r}^{n} = F_{r+1}^{n-1}. \tag{10}$$

In particular,

$$\Delta F_0^n = F_1^n - F_0^n = F_1^{n-1}$$
$$\Delta^2 F_0^n = \Delta F_1^{n-1} = F_2^{n-2}$$

and, in general,

$$\Delta^t F_0^n = F_t^{n-t}. \tag{11}$$

Thus, the leading term and leading differences of the series of figurate numbers F_r^n, $(r=0, 1, 2, \cdots)$, are the figurate numbers F_0^n, F_1^{n-1}, F_2^{n-2}, *etc.*, which, as in Table 1, form a downward diagonal to the left, starting with F_0^n. It may be observed that the numbers above such a diagonal do not appear at all when a series of figurate numbers is successively differenced by the repeated operation $\Delta u_x = u_{x+1} - u_x$. Apparently the operation of inverse summation, which we shall represent by the symbol S^{-1} (corresponding to S for summation), is a more general type of difference operation since it produces the numbers above the leading difference diagonal in addition to the leading differences themselves. This suggests the desirability of studying similar patterns derived from other series.

3. Summation and Inverse Summation of Series. When generalized figurate numbers are arranged as in Table 1 any two adjacent columns bear mutually inverse relationships to each other. The series at the right is the summation of the series at the left. The series at the left is the inverse summation of the series at the right. In effect, any one of the columns may be regarded as a generating series from which all of the others are derived.

Analogously, the successive equidistant values of a function u_x, $(x=0, 1, 2, \cdots, r)$, may be treated as a generating series and operated upon by repeated summation and by repeated inverse summation. If u_r represents the general term of the generating series, Su_r the general term of the summation series, and $S^{-1}u_r$ the general term of the inverse summation series, the following relationships are true:

$$S^0 u_r = u_r \tag{12}$$

$$Su_r = u_0 + u_1 + u_2 + \cdots + u_r = \sum_{t=0}^{r} u_t \tag{13}$$

$$S^2 u_r = Su_0 + Su_1 + Su_2 + \cdots + Su_r = \sum_{t=0}^{r} Su_t, \quad etc. \tag{14}$$

(15) $S^{-1}u_r = u_r - u_{r-1}, \qquad r > 0$

(16) $S^{-2}u_r = S^{-1}u_r - S^{-1}u_{r-1}, \qquad r > 0, \quad etc.$

In particular,

(17) $S^{-1}u_1 = u_1 - u_0 = \Delta u_0$

(18) $S^{-2}u_2 = S^{-1}u_2 - S^{-1}u_1 = \Delta S^{-1}u_1 = \Delta^2 u_0.$

Analogous to (11)

(19) $\Delta^t S^n u_0 = S^{n-t}u_t.$

When $r=0$, no operation of summation or of inverse summation is performed on u_0. Therefore, analogous to (3),

(20) $S^m u_0 = S^{-n}u_0 = u_0.$

Table 2 exhibits basic relationships between the generating series, the summation series and the inverse summation series. It should be noted that the

TABLE 2

x	*Inverse Summation* $S^{-1}u_x$	*Generating Series* S^0u_x	*Summation* Su_x
0	u_0	u_0	u_0
1	u_1-u_0	u_1	u_0+u_1
...	...	...	...
r	u_r-u_{r-1}	u_r	$u_0+u_1+\cdots+u_r$

mutually inverse operations S and S^{-1} are commutative. By contrast, the operation Δ and its inverse Δ^{-1} (or Σ) are not commutative [2].

4. Parallel Leading Differences and Intersecting Leading Differences. Suppose u_x is a rational integral function of the nth degree in x, and that the series of equidistant terms u_x, $(x=0, 1, \cdots, n+1)$, has been operated on by repeated inverse summation. Table 3 summarizes certain results of such operations; that is, it lists parallel leading differences and intersecting leading differences of a

TABLE 3

x	$S^{-(n+1)}u_x$	$S^{-n}u_x$	$S^{-(n-1)}u_x$	...	$S^{-2}u_x$	$S^{-1}u_x$	S^0x_x
0	u_0	u_0	u_0	...	u_0	u_0	u_0
1	d_1		...			Δu_0	u_1
2	d_2		...		$\Delta^2 u_0$		u_2
...	...		...	...			...
$n-1$	d_{n-1}		$\Delta^{n-1}u_0$		...		u_{n-1}
n	d_n	$\Delta^n u_0$			...		u_n
$n+1$	0	$\Delta^n u_0$			...		u_{n+1}
...	0	...			...		...

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rational integral function of the nth degree in x. In column $S^{-n}u_x$ the nth differences, starting with $\Delta^n u_0$, are constant. In the successive columns starting with $S^{-(n+1)}u_x$ the $(n+1)$th and higher orders of differences vanish. In general, the terms $d_1, d_2, \cdots, d_n$ in column $S^{-(n+1)}u_x$ are not zero. We designate them the parallel leading differences of the generating series, in order to distinguish them from $\Delta u_0, \Delta^2 u_0, \cdots, \Delta^n u_0$ which we call the intersecting leading differences. In this way, both sets of differences may be derived by repeated inverse summation of the $(n+1)$th order.

The rth parallel leading difference is

$$d_r = S^{-(n+1)}u_r. \tag{21}$$

The rth intersecting leading difference is

$$\Delta^r u_0 = S^{-r}u_r. \tag{22}$$

5. Criss-Cross Multiplication of Series. Having given two series a_x and b_x, $(x=0, 1, \cdots, r)$, a third series $c_x=a_x*b_x$ can be obtained by a process of criss-cross multiplication shown in Table 4. The symbol of criss-cross multiplication

TABLE 4

x	a_x	b_x	$c_x=a_x*b_x$
0	a_0	b_0	$c_0=a_0b_0$
1	a_1	b_1	$c_1=a_0b_1+a_1b_0$
...	...	...	...
r	a_r	b_r	$c=a_r*b_r=\sum_{t=0}^{r} a_t b_{r-t}$

$(*)$ is somewhat suggestive of the operation which it represents. The symbol is taken from the theory of the Laplace Transformation where it has an analogous significance in connection with certain continuous functions [3]. The operation of criss-cross multiplication obviously conforms to the commutative law; that is, $a_r*b_r=b_r*a_r$. It is easily seen that the operation of criss-cross multiplication conforms to the distributive law; that is, $a_r*(b_r+c_r)=a_r*b_r+a_r*c_r$. We shall now demonstrate that it also conforms to the associate law.

Proof:

$$(a_r*b_r)*c_r = (a_0*b_0)c_r + (a_1*b_1)c_{r-1} + \cdots + (a_{r-1}*b_{r-1})c_1 + (a_r*b_r)c_0. \tag{23}$$

Multiplying out and regrouping the terms, we obtain

$$(a_r*b_r)*c_r = a_0(b_r*c_r) + a_1(b_{r-1}*c_{r-1}) + \cdots + a_r(b_0*c_0) \tag{24}$$

$$= a_r*(b_r*c_r). \tag{25}$$

$$(a_r*b_r)*c_r = b_0(a_r*c_r) + b_1(a_{r-1}*c_{r-1}) + \cdots + b_r(a_0*c_0) \tag{26}$$

$$= b_r*(a_r*c_r). \tag{27}$$

6. Theorems on Criss-Cross Multiplication, Figurate Numbers, and Summation of Series.

THEOREM 1. *Repeated summation of a series n times is equivalent to criss-cross multiplication of the series by the nth order of figurate numbers.* That is,

$$S^n u_r = u_r * F_r^n \tag{28}$$

$$= u_0 F_r^n + u_1 F_{r-1}^n + \cdots + u_{r-1} F_1^n + u_r F_0^n \tag{29}$$

$$= u_0 \binom{n+r-1}{r} + u_1 \binom{n+r-2}{r-1} + \cdots + u_{r-1} \binom{n}{1} + u_r \binom{n-1}{0} \tag{30}$$

$$= u_0 \binom{n+r-1}{n-1} + u_1 \binom{n+r-2}{n-1} + \cdots + u_{r-1} \binom{n}{n-1} + u_r \binom{n-1}{n-1}. \tag{31}$$

An equivalent result has been given by Sheppard [4] with a reference to Elderton [5].

THEOREM 2. *Repeated inverse summation of a series n times is equivalent to criss-cross multiplication of the series by the $(-n)$th order of figurate numbers. That is,*

$$S^{-n} u_r = u_r * F_r^{-n} \tag{32}$$

$$= u_0 F_r^{-n} + u_1 F_{r-1}^{-n} + \cdots + u_{r-1} F_1^{-n} + u_r F_0^{-n} \tag{33}$$

$$= u_0 (-1)^r \binom{n}{r} + u_1 (-1)^{r-1} \binom{n}{r-1} + \cdots + u_r \binom{n}{0} \tag{34}$$

$$= u_r - \binom{n}{1} u_{r-1} + \binom{n}{2} u_{r-2} - \cdots + (-1)^r \binom{n}{r} u_0. \tag{35}$$

An equivalent result has been given by Dwyer [6].

THEOREM 3. *Criss-cross multiplication of figurate numbers of order r (positive or negative) by figurate numbers of order s (positive or negative) results in figurate numbers of order $r+s$.*

(36) That is, $F_m^r * F_m^s = F_m^{r+s}$

(37) Corollary: $F_m^r * F_m^s * F_m^{-t}$ *etc.* $= F_m^{r+s-t \text{ etc.}}$.

THEOREM 4. *If $u_x(x=0, 1, 2, \cdots, n)$ is a rational integral function of the nth degree in x, the rth parallel leading difference is*

$$d_r = S^{-(n+1)}u_r = u_r * F_r^{-(n+1)} \tag{38}$$

and the rth intersecting leading difference is

$$\Delta^r u_0 = S^{-r}u_r = u_r * F_r^{-r} = d_r * F_r^{n-r+1}. \tag{39}$$

THEOREM 5. *Let u_x and v_x represent the successive terms of two series for $x=0, 1, 2$, etc., and let $u_r * v_r$ represent the general term of their criss-cross product. If the first series is operated on by repeated summation of the mth order, and the second series by repeated inverse summation of the nth order, then the general term of the criss-cross product of the two new series is*

$$S^m u_r * S^{-n} v_r = (F_r^m * u_r) * (F_r^{-n} * v_r) = F_r^{m-n} * (u_r * v_r) = S^{m-n}(u_r * v_r). \tag{40}$$

THEOREM 6. *If in the preceding formula $n=m$, it becomes*

$$S^m u_r * S^{-m} v_r = F_r^0 * (u_r * v_r) = u_r * v_r. \tag{41}$$

This last result is equivalent to Dwyer's Successive Cumulation Theorem [6], which he applied to the development of new techniques for the computation of moments of statistical frequency distributions.

Proofs of theorems 1 to 6 present no unusual features, and are omitted here.

7. Applications. To derive a general formula for the $(r+1)$th term in the kth order of repeated summation of the series $u_x(x=0, 1, \cdots, r)$, a rational integral function of the nth degree in x:

In Table 3, any column operated upon by repeated summation and inverse summation will produce the entire field of numbers. Therefore, any column operated upon by criss-cross multiplication with figurate numbers will produce the entire field. If we choose the column which contains the parallel leading differences of the original series, that is, if we apply criss-cross multiplication by figurate numbers to the series $u_0, d_1, d_2, \cdots, d_n$, then any term in any column can be represented by an expression of not more than $n+1$ terms involving the parallel leading differences and figurate numbers.

Applying the foregoing principles, the desired term is $S^k u_r$, and

$$S^k u_r = u_r * F_r^k = (d_r * F_r^{n+1}) * F_r^k = d_r * F_r^{k+n+1} \tag{42}$$

$$= u_0 F_r^{k+n+1} + d_1 F_{r-1}^{k+n+1} + \cdots + d_n F_{r-n}^{k+n+1} \tag{43}$$

$$= u_0 \binom{k+n+r}{r} + d_1 \binom{k+n+r-1}{r-1} + \cdots + d_n \binom{k+r}{r-n} \tag{44}$$

$$= u_0 \binom{k+n+r}{k+n} + d_1 \binom{k+n+r-1}{k+n} + \cdots + d_n \binom{k+r}{k+n}. \tag{45}$$

The parallel leading differences d_1, d_2, *etc.* which are required in applying the above formulas may be obtained either by repeated inverse summation or by criss-cross multiplication of the given series by the series of figurate numbers of order $-(n+1)$, that is, by $F^{-(n+1)}$. It may be observed that the combinatorial expressions represent figurate numbers of constant order $(k+n+1)$. Such expressions are simpler to evaluate numerically than the binomial coefficients of constant order which appear in corresponding formulas in terms of the intersecting leading differences Δ, Δ^2, Δ^3, *etc.*

8. Numerical Examples. A few numerical examples will illustrate the application of the summation formula described in the preceding section.

Example (a): Derive an expression for the sum of the cubes of the first r odd numbers. Here, $u_x=(2x+1)^3$, the expression is of the third degree, and the parallel leading differences may be obtained by criss-cross multiplication by figurate numbers of the fourth negative order, that is, by F_x^{-4}, as follows:

x	$u_x=(2x+1)^3$	F_x^{-4}	$d_x=u_x * F_x^{-4}$
0	1	1	1
1	27	−4	23
2	125	6	23
3	343	−4	1

In this case, $k=1$; $n=3$; $u_0=1$; $d_1=23$; $d_2=23$; $d_3=1$; and the required summation ends with the term u_{r-1}. Substituting,

$$Su_{r-1} = \binom{r+3}{4} + 23\binom{r+2}{4} + 23\binom{r+1}{4} + \binom{r}{4}$$

$$= \frac{(r+3)(r+2)(r+1)r}{24} + \frac{23(r+2)(r+1)r(r-1)}{24}$$

$$+ \frac{23(r+1)r(r-1)(r-2)}{24} + \frac{r(r-1)(r-2)(r-3)}{24}.$$

When this is multiplied out, it reduces to

$$1^3 + 3^3 + \cdots + (2r-1)^3 = r^2(2r^2-1).$$

Example (b): Find polynomial expressions for the rth term and the sum of r terms of the series $-6, -3, 10, 39, 90, 169, \cdots$.

Since, in this case, inspection does not reveal the degree of the general term, the parallel differences are formed by repeated inverse summation, as follows:

x	u_x	$S^{-1}u_x$	$S^{-2}u_x$	$S^{-3}u_x$	$S^{-4}u_x=d_x$
0	−6	−6	−6	−6	−6
1	−3	3	9	15	21
2	10	13	10	1	−14
3	39	29	16	6	5
4	90	51	22	6	0
5	169	79	28	6	0

From the above difference table, since terminal zeros appear in column $S^{-4}u_x$, we assume that the desired polynomial is of the third degree. The term desired is u_{r-1}, since that is the rth term of the series $u_0, u_1, u_2, \cdots, u_{r-1}$. Substituting in formula (45), $k=0$; $n=3$; $u_0=-6$; $d_1=21$; $d_2=-14$; $d_3=5$; and for the rth term of the series

$$S^0u_{r-1} = -6\binom{r+2}{3} + 21\binom{r+1}{3} - 14\binom{r}{3} + 5\binom{r-1}{3}$$

$$= r^3 - r^2 - r - 5.$$

Similarly, for the sum of the first r terms,

$$Su_{r-1} = -6\binom{r+3}{4} + 21\binom{r+2}{4} - 14\binom{r+1}{4} + 5\binom{r}{4}$$

$$= r(3r^3 + 2r^2 - 9r - 68)/12.$$

9. Vandermonde's Theorem. Using the notation $r^{(k)}=r(r-1)\cdots(r-k+1)$, the identity known as Vandermonde's Theorem or the factorial binomial theorem [7] may be written

$$(r+s)^{(k)} = r^{(k)} + \binom{k}{1}r^{(k-1)}s + \binom{k}{2}r^{(k-2)}s^{(2)} + \cdots \tag{46}$$

which is analogous in form to the binomial expansion

$$(r+s)^k = r^k + \binom{k}{1}r^{k-1}s + \binom{k}{2}r^{k-2}s^2 + \cdots. \tag{47}$$

The factorial binomial theorem may be proved starting with Theorem 3, as follows:

$$F_k^{-r-s} = F_k^{-r} * F_k^{-s} \tag{48}$$

$$(49)\quad (-1)^k\binom{r+s}{k} = F_k^{-r}F_0^{-s} + F_{k-1}^{-r}F_1^{-s} + \cdots + F_0^{-r}F_k^{-s}$$

$$= (-1)^k\binom{r}{k}\binom{s}{0} + (-1)^{k-1}\binom{r}{k-1}(-1)^1\binom{s}{1}$$

$$+ \cdots + \binom{r}{0}(-1)^k\binom{s}{k}.$$

Eliminating $(-1)^k$ from both sides of the equation,

$$(50)\quad \binom{r+s}{k} = \binom{r}{k}\binom{s}{0} + \binom{r}{k-1}\binom{s}{1} + \cdots + \binom{r}{0}\binom{s}{k}, \text{ or}$$

$$(51)\quad \frac{(r+s)^{(k)}}{k!} = \frac{r^{(k)}s^{(0)}}{k!0!} + \frac{r^{(k-1)}s^{(1)}}{(k-1)!1!} + \cdots + \frac{r^{(0)}s^{(k)}}{0!k!}.$$

Multiplying both sides by $k!$ we obtain

$$(52)\quad (r+s)^{(k)} = r^{(k)}s^{(0)} + \binom{k}{1}r^{(k-1)}s^{(1)} + \binom{k}{2}r^{(k-2)}s^{(2)} + \cdots + r^{(0)}s^{(k)}.$$

10. A Problem in Binomial Coefficients. The properties of generalized figurate numbers can be applied to solve Problem 4189 in the MONTHLY, February, 1946, Vol. 53, No. 2, page 103.
Prove that

$$\sum_{r=0}^{m}(-1)^r\binom{m+k+a}{r+k+a}\binom{r+k}{k} = \binom{m+a-1}{m}, \qquad a \geqq 0.$$

The expression

$$(-1)^r\binom{m+k+a}{r+k+a} = (-1)^r\binom{m+k+a}{m-r} = (-1)^m F_{m-r}^{-(m+k+a)}.$$

The expression

$$\binom{r+k}{k} = \binom{r+k}{r} = F_r^{k+1}.$$

Therefore, the original expression is equivalent to

$$(-1)^m \cdot \sum_{r=0}^{m} F_{m-r}^{-(m+k+a)} \cdot F_r^{k+1} = (-1)^m F_m^{-(m+k+a)} * F_m^{k+1}$$

$$= (-1)^m F_m^{-(m+a-1)} = \binom{m+a-1}{m}.$$

The result is independent of the restriction $a \geqq 0$. The above proof may be compared with the proof in the MONTHLY, June–July, 1947, page 347.

11. Summation of Product Series. Theorem 6 can be applied to find the sum of a product series such as $u_r v_0, u_{r-1} v_1, \cdots, u_0 v_r$, where v_x is a polynomial of degree n. The sum of the series is evidently

$$u_r * v_r = u_r v_0 + u_{r-1} v_1 + \cdots + u_0 v_r. \tag{53}$$

Introducing figurate numbers of the $(n+1)$th order, positive and negative,

$$u_r * v_r = (F_r^{n+1} * u_r) * (F_r^{-(n+1)} * v_r) \tag{54}$$

$$= (F_r^{n+1} * u_r) * d_r. \tag{55}$$

Changing back from criss-cross multiplication by figurate numbers to repeated summation,

$$u_r * v_r = S^{n+1} u_r * d_r, \tag{56}$$

When the above is expanded, since the function v_x is of the nth degree, the terms involving d_{n+1}, d_{n+2}, *etc.*, will vanish. Therefore, the desired sum

$$u_r * v_r = d_0 S^{n+1} u_r + d_1 S^{n+1} u_{r-1} + \cdots + d_n S^{n+1} u_{r-n}, \tag{57}$$

where

$$d_t = F_t^{-(n+1)} * v_t, \qquad (t = 0, 1, \cdots, n).$$

This result is equivalent to one given by Dwyer [6] for the evaluation of $\sum_0^k u_{a+x} v_{a+x}$.

12. Parallel Leading Differences of Powers of Integers. Although Dwyer used another type of notation, the differencing operation which he applied is identical with the operation of inverse summation as defined in this article. Carrying out his objective of simplifying the computation of statistical moments, Dwyer derived general expressions and recurrence formulas for what, in our terminology, are the parallel leading differences of the powers of integers, which are analogous to the "differences of zero" of actuarial literature. For the powers r^n, $(r = 0, 1, 2, \cdots)$, since the function is of the nth degree, Theorem 4 is applicable, and we have

$$d_r = S^{-(n+1)} r^n = F_r^{-(n+1)} * r^n. \tag{58}$$

An equivalent formula, in another connection has been given by Wall [8]. Values of $d_1, d_2, \cdots$ are given in Table 5 from $n = 1$ to $n = 11$.

TABLE 5

r	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$	$n=11$
1	1	1	1	1	1	1	1	1	1	1	1
2		1	4	11	26	57	120	247	502	1013	2036
3			1	11	66	302	1191	4293	14608	47840	152637
4				1	26	302	2416	15619	88234	455192	2203488
5					1	57	1191	15619	156190	1310354	9738114
6						1	120	4293	88234	1310354	15724248
7							1	247	14608	455192	9738114
8								1	502	47840	2203488
9									1	1013	152637
10										1	2036
11											1
Totals	1	2	6	24	120	720	5040	40320	362880	3628800	39916800
	1!	2!	3!	4!	5!	6!	7!	8!	9!	10!	11!

These values of d may be substituted in Formula 45 to derive expressions for powers of integers and their sums in terms of figurate numbers or in terms of factorial products. From a statistical point of view, r^n turns out to be a weighted average of factorial products, in which the d's are the weights. Thus,

$$r^n = \frac{d_1 r^{(n)} + d_2 (r+1)^{(n)} + \cdots + d_n (r+n-1)^{(n)}}{d_1 + d_2 + \cdots + d_n}. \tag{59}$$

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MATHEMATICAL NOTES

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ON THE HARMONIC MEAN OF TWO NUMBERS

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We know that x satisfies the double inequality

$$9 \leqq x \leqq 11,$$

but we know nothing else about the value of x. In such a case, we should take for x the approximate value 9.9; then we can be certain that the relative error cannot exceed 10% and this is the best we can get. The following lines explain this example and prove the underlying theorem.

About an otherwise unknown quantity x we have just one piece of definite information: we know that x is contained between two given positive bounds a and b,

$$a \leqq x \leqq b, \tag{1}$$

where

$$0 < a < b. \tag{2}$$

In choosing a proximate value p for x we risk a certain *error* $p-x$ and a certain *relative error* $(p-x)/x$. We may wish to reduce the risk of an extreme error, or the risk of an extreme relative error, to a minimum; and so two different problems arise.

PROBLEM I. *Find*

$$\min_{p}\left(\max_{x} \mid p - x \mid\right)$$

and the value of p for which it is attained under the conditions (1) *and* (2).

PROBLEM II. *Find*

$$\min_{p}\left(\max_{x} \mid p - x \mid / x\right)$$

and the value of p for which it is attained under the conditions (1) *and* (2).

The solution of problem I is obvious and certainly known to many persons although I am not able to give a definite quotation:

$$p = \frac{a+b}{2}, \qquad \min_{p} \max_{x} |p - x| = \frac{b-a}{2}.$$

The solution of problem II is a little less obvious and may be new, but is certainly very little known:

$$p = \frac{2ab}{a+b}, \qquad \min_{p} \max_{x} \frac{|p-x|}{x} = \frac{b-a}{b+a}.$$

We can state the second result in words as follows: *The approximation that yields the minimum for the greatest possible absolute value of the relative error, committed in approximating an unknown quantity contained between two known positive bounds, is the harmonic mean of these bounds.*

In both problems we can find the desired p by the same prescription: Choose p so that the errors (relative errors) committed in the two extreme admissible cases, $x=a$ and $x=b$, are equal in amount, but opposite in sign. This yields

$$(3) \qquad p - a = -(p - b), \qquad p = (a+b)/2, \qquad p - a = (a+b)/2,$$

$$(4) \qquad \frac{p-a}{a} = -\frac{p-b}{b}, \qquad p = \frac{2ab}{a+b}, \qquad \frac{p-a}{a} = \frac{b-a}{b+a},$$

in problems I and II, respectively. Following this prescription, we are bound to obtain the best possible values, by virtue of a classical theorem of Tchebischeff.[1] Our problems are so simple, however, that the result can be proved independently of any special knowledge. I give the proof for the solution of the less obvious problem II.

We regard the approximate value p as chosen. We focus our attention on the more interesting case in which $a \leqq p \leqq b$. The absolute value of the relative error depends on x:

$$(5) \qquad \frac{|p-x|}{x} = \begin{cases} \dfrac{p}{x} - 1 & \text{when} \quad a \leqq x \leqq p \\ 1 - \dfrac{p}{x} & \text{when} \quad p \leqq x \leqq b. \end{cases}$$

When x varies from a to b, (5) first decreases from $(p-a)/a$ to 0 then increases from 0 to $(b-p)/b$. Therefore, the maximum of (5) must be attained at one of the two end-points, at $x=a$ or at $x=b$. Now,

$$(6) \qquad \frac{a}{a+b}\,\frac{p-a}{a} + \frac{b}{a+b}\,\frac{b-p}{b} = \frac{b-a}{b+a}.$$

The right-hand side of (6) is a weighted mean of $(p-a)/a$ and $(b-p)/b$. If these

[1] See *e.g.* Serge Bernstein, Leçons sur les propriétés extrémales des fonctions analytiques d'une variable réelle, Paris, 1926, pp. 2–4.

quantities are *different*, one of them is *greater* than the right-hand side of (6), that is

$$\max_{x} \frac{|p-x|}{x} > \frac{b-a}{b+a}. \tag{7}$$

It happens only in the case (4) that (7) goes over into an equation. This proves our theorem in the case in which the value of p lies between a and b. The two remaining (*a priori* unplausible) cases, $p<a$ and $p>b$, can be discussed even more simply.

NOTE ON CAUCHY'S LIMIT THEOREM

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Cauchy's limit theorem states that $\lim_{n\to\infty}(a_1+\cdots+a_n)/n=\lim_{n\to\infty} a_n$ if the second limit exists. We may generalize this theorem as follows.

THEOREM 1. *Let a_n be a sequence of real numbers convergent to A: $a_n\to A$. Let $f_n(\alpha_1, \alpha_2, \alpha_3, \cdots)$ be a sequence of real-valued functions which is defined for any such sequence $(\alpha_1, \alpha_2, \alpha_3, \cdots)$ that, for some positive integer h, $\alpha_\nu=a_\nu$ or A when $\nu\leqq h$ and $\alpha_\nu=a_\nu$ when $\nu>h$. Suppose that such individual a_k is ultimately immaterial to $f_n(a_1, a_2, a_3, \cdots)$ and $f_n(a_1, a_2, a_3, \cdots)$ is ultimately a mean, namely that*

$$\lim_{n\to\infty}\left[f_n(\alpha_1, \cdots, \alpha_{k-1}, \alpha_k, a_{k+1}, a_{k+2}, \cdots) - f_n(\alpha_1, \cdots, \alpha_{k-1}, A, a_{k+1}, a_{k+2}, \cdots)\right] = 0 \tag{1}$$

and

$$\min(\alpha_1, \cdots, \alpha_k, a_{k+1}, a_{k+2}, \cdots) + \eta_n \leqq f_n(\alpha_1, \cdots, \alpha_k, a_{k+1}, a_{k+2}, \cdots) \leqq \max(\alpha_1, \cdots, \alpha_k, a_{k+1}, a_{k+2}, \cdots) + \zeta_n \tag{2}$$

for every $(\alpha_1, \alpha_2, \alpha_3, \cdots)$ and for every positive integer k, where $\eta_n\to 0$, $\zeta_n\to 0$. Then

$$f_n(a_1, a_2, a_3, \cdots)\to A.$$

For we may write

$$\begin{aligned} f_n(a_1, a_2, a_3, \cdots) - A = {} & f_n(a_1, a_2, a_3, \cdots) - f_n(A, a_2, a_3, \cdots) \\ & + f_n(A, a_2, a_3, \cdots) - f_n(A, A, a_3, \cdots) \\ & \vdots \\ & + f_n(A, \cdots, A, a_k, a_{k+1}, \cdots) \\ & - f_n(A, \cdots, A, a_{k+1}, a_{k+2}, \cdots) \\ & + f_n(A, \cdots, A, a_{k+1}, a_{k+2}, \cdots) - A, \end{aligned}$$

hence (1) implies that

$$\overline{\lim_{n\to\infty}} \left| f_n(a_1, a_2, a_3, \cdots) - A \right| \leqq \overline{\lim_{n\to\infty}} \left| f_n(A, \cdots, A, a_{k+1}, a_{k+2}, \cdots) - A \right|.$$

And (2) implies that the right member of the above inequality tends to zero as $k\to\infty$. Thus the left member of the inequality is equal to zero, and the theorem is proved.

Similarly we have the following result.

THEOREM 2. *Let a_n be a sequence of complex numbers convergent to A, and let $f_n(\alpha_1, \alpha_2, \alpha_3, \cdots)$ be a sequence of complex-valued functions which is defined for any such sequence $(\alpha_1, \alpha_2, \alpha_3, \cdots)$ that, for some positive integer h, $\alpha_\nu = a_\nu$ or A when $\nu \leqq h$, and $\alpha_\nu = a_\nu$ for $\nu > h$ or $\alpha_\nu = A$ for $\nu > h$. Suppose that each individual a_k is ultimately immaterial to $f_n(a_1, a_2, a_3, \cdots)$ and $f_n(\alpha_1, \alpha_2, \alpha_3, \cdots)$ is ultimately continuous, namely that condition* (1) *is satisfied together with the condition*

$$\lim_{k\to\infty} \overline{\lim_{n\to\infty}} \left| f_n(\alpha_1, \cdots, \alpha_k, a_{k+1}, a_{k+2}, \cdots) - f_n(\alpha_1, \cdots, \alpha_k, A, A, \cdots) \right| = 0 \tag{2'}$$

for each $(\alpha_1, \alpha_2, \alpha_3, \cdots)$ and k. Then

$$f_n(a_1, a_2, a_3, \cdots) = f_n(A, A, A, \cdots) + o(1).$$

It is to be observed that conditions (1) and (2) are also *necessary* for $a_n\to A$ to imply $f_n(a_1, a_2, a_3, \cdots)\to A$; and that (1) and (2′) form also necessary conditions for $a_n\to A$ to imply the conclusion of Theorem 2.

The necessity of (1) is evident; for, if $(a_1, a_2, a_3, \cdots)$ has limit A then $(\alpha_1, \cdots, \alpha_{k-1}, A, a_{k+1}, a_{k+2}, \cdots)$ has the same limit. And (2) follows immediately from the relations:

$$(\alpha_1, \cdots, \alpha_k, a_{k+1}, a_{k+2}, \cdots) \to A, \qquad f_n(\alpha_1, \cdots, \alpha_k, a_{k+1}, a_{k+2}, \cdots) \to A.$$

A similar argument proves the necessity of (2′).

We note also that Theorems 1 and 2 hold for $A = \pm\infty$ as well as for finite A if we substitute A_n for A in conditions (1), (2), and in the conclusion of Theorem 2 together with the definition of $f_n(\alpha_1, \alpha_2, \alpha_3, \cdots)$, where A_n is any sequence (real in the case of Theorem 1) such that $A_n\to A$. The necessity of the new conditions thus obtained is still only for finite A, however.

The well-known generalizations (due to Jensen, Stolz, Toeplitz and Silverman) of Cauchy's theorem lay stress on the linear form of $f_n(a_1, a_2, a_3, \cdots)$. (See K. Knopp, *Theory and Application of Infinite Series*, English translation by R. C. Young, pp. 70–76.) The above extensions contain these as particular cases and seem sometimes more convenient to apply.

Silverman's theorem reads: If

$$a_n \to 0 \quad \text{or} \quad \sum_{\nu=1}^{\infty} t_{n\nu} \to 1,$$

then

$$\sum_{\nu=1}^{\infty} t_{n\nu}a_\nu \to A,$$

provided that $a_n \to A$ and that

$$(a) \qquad t_{nk} = o(1) \qquad \text{for } k = 1, 2, 3, \cdots,$$

$$(b) \qquad \sum_{\nu=1}^{\infty} |t_{n\nu}| = O(1).$$

This immediately follows from Theorem 2, since (a) implies (1), (b) implies $(2')$.

By making use of Theorem 2, it is also readily seen that a generalization of Knopp's theorem on bilinear form (*op. cit.* p. 73) runs as follows.

If $(p_{n1}, p_{n2}, p_{n3}, \cdots)$ *is a permutation of* $(1, 2, 3, \cdots)$ *for each* n, *then*

$$\sum_{\nu=1}^{\infty} t_{n\nu}a_\nu b_{p_{n\nu}} \to AB \quad \textit{when} \quad a_n \to A,\ b_n \to B\ (A, B \textit{ finite}),$$

provided that $\sum_{\nu=1}^{\infty} t_{n\nu} \to 1$, *and that* (a) *and* (b) *are satisfied together with the condition*

$$(a') \qquad \lim_{n\to\infty} t_{n\nu_{nk}} = 0 \qquad \textit{for } k = 1, 2, 3, \cdots,$$

where ν_{nk} *is the integer value of* ν *for which* $p_{n\nu} = k$.

For, (b) implies that the function

$$f_n(a_1, a_2, a_3, \cdots) = \sum_{\nu=1}^{\infty} t_{n\nu}a_\nu b_{p_{n\nu}}$$

is ultimately continuous, and (a) implies that each a_k is immaterial to it; hence we have

$$f_n(a_1, a_2, a_3, \cdots) = \sum_{\nu=1}^{\infty} t_{n\nu}Ab_{p_{n\nu}} + o(1) = Ag_n(b_1, b_2, b_3, \cdots) + o(1),$$

where

$$g_n(b_1, b_2, b_3, \cdots) = \sum_{\nu=1}^{\infty} t_{n\nu}b_{p_{n\nu}}.$$

Now (b) implies that $g_n(b_1, b_2, b_3, \cdots)$ is ultimately continuous and (a') implies that each b_k is immaterial to it; hence we also have

$$g_n(b_1, b_2, b_3, \cdots) = \sum_{\nu=1}^{\infty} t_{n\nu}B + o(1) = B + o(1).$$

Therefore $f_n(a_1, a_2, a_3, \cdots) = AB + o(1)$, and the statement is proved.

To illustrate Theorem 1, let a_{ij}^n $(i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, n)$ be new

notations for $a_1, a_2, a_3, \cdots, a_{n^2}$ in any order, and let

$$|\overset{n}{a_{ij}}|^+ = \sum_p \overset{n}{a_{1p_1}}\overset{n}{a_{2p_2}} \cdots \overset{n}{a_{np_n}},$$

where the summation extends over all the permutations of $(1, 2, \cdots, n)$. Then we have

$$\frac{1}{n}(|\overset{n}{a_{ij}}|^+)^{1/n} \to \frac{1}{e}A \qquad \text{if } a_n \to A,\ a_n \geqq 0.$$

For, Theorem 1 (with $\eta_n = \zeta_n = 0$) gives

$$\left(\frac{1}{n!}|\overset{n}{a_{ij}}|^+\right)^{1/n} \to A,$$

and

$$\lim_{n\to\infty}\left(\frac{1}{n!}\right)^{1/n}(n+1) = \lim_{n\to\infty}\left[\left(\frac{2}{1}\right)^1\left(\frac{3}{2}\right)^2\cdots\left(\frac{n+1}{n}\right)^n\right]^{1/n}$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = e.$$

Finally, the author should like to express his indebtedness and gratitude to the referee for invaluable criticisms and suggestions.

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College

After February 1, 1950 all material for this department should be sent to C. B. Allendoerfer, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

A PARADOX RELATING TO MATHEMATICAL INDUCTION

R. G. ALBERT, Providence, R. I.

The axiom of mathematical induction as included in the set of axioms of Peano which characterize the system of natural numbers asserts:

If S be a set of natural numbers containing 1 and containing the successor x' of every natural number x which S contains, then S is the set of *all* natural numbers.

The following proof by mathematical induction (by Landau), that every natural number except 1 has an immediate predecessor, suggested an interesting

paradox which may aid in gaining an insight into this important method of proof:

Let x' denote the successor of x (i.e., $x+1$).

Let S be the set of natural numbers including 1 and including all n.n.'s u such that there exists a n.n. x with $x'=u$. Then, obviously, S contains 1. If S contains $u=k$, then S contains $u=k'$, since $x'=k'$ has the evident solution $x=k$. Hence S is the set of all n.n.'s. Since S was defined to be the set comprised of 1 and of all n.n.'s u which have predecessors, it follows that every n.n. except 1 has an immediate predecessor.

Now, examine the following proof which seems to trace the same lines:

We shall try to show that every n.n. except 1 equals its own successor.

Let S be the set of n.n.'s containing 1 and containing all n.n.'s u such that $u=u'$ (u' denotes $u+1$, as before). S contains 1. If S contains $u=k$, then $k=k'$. Since every n.n. has a unique successor (Axiom 2, Peano), then $k'=(k')'$. The latter verifies $u=u'$ for $u=k'$. Thence, S contains $u=k'$. Thus, S is the set of all n.n.'s. Since S was defined to contain 1 and all n.n.'s equal to their own successors, it follows that every n.n. with the possible exception of 1 is its own successor.

This procedure is of course generalizable and seems to suggest that many theorems or formulas which are known to be false can be "proved" for all n.n.'s except 1, by this device of consigning 1 to S a priori. It appears as if tossing 1 in bodily from the start is a kind of circumvention of the intention in the induction hypothesis, raising doubts regarding the validity of arbitrary inclusion of 1.

It has been of interest to me to note how many of my students have failed to "see through" the error in the last proof and why the first proof is still valid.

Of course, the error in the second demonstration lies in the step:

If S contains $u=k$, then $k=k'$.

From S containing $u=k$, follows: *Either* $k=1$ *or* $k=k'$. And in the first of these alternative cases, we can no longer assert that S contains k', since neither qualification for admission to S is satisfied by k', i.e., $1'\neq 1$ and $1'\neq(1')'$.

In the first proof, S containing k *did* imply *invariably* that S contained k', for $x'=k'$ does in all cases have a solution, $x=k$.

A NOTE ON TAYLOR'S THEOREM

C. L. Seebeck, Jr., University of Alabama

Although there are many proofs of Taylor's Theorem in mathematical literature, most of these are difficult for a not-too-advanced Calculus student to master. The following proof requires no knowledge of infinite series, or of convergence. It is direct, readily illustrated geometrically, and provides the remainder using only elementary or easily explained theorems.

Let $f(x)$ be a function continuous with its derivatives through order $n+1$ in a suitable neighborhood of $x=a$. Moreover, assume $f^{[k]}(a)$ may be evaluated for

$k=0, 1, 2, \cdots, n$. We desire the value of $f(x)$ at a point in the given neighborhood. Since the evaluation process may call for operations other than $(+, -, \times, \div)$, we first approximate its value by means of

$$P(x) = \sum_{k=0}^{n} c_k(x-a)^k,$$

a polynomial. We choose c_k so that $P(x)$ will be identified with $f(x)$ as closely as possible when $x=a$, or such that

$$P^{[k]}(a) = f^{[k]}(a), \qquad k = 0, 1, 2, \cdots, n, \tag{1}$$

a condition that implies

$$c_k = \frac{f^{[k]}(a)}{k!}.$$

Now let $R(x)$ be the amount by which the approximation fails.

$$R(x) = f(x) - P(x). \tag{2}$$

Equation (1) implies

$$R^{[k]}(a) = 0, \qquad k = 0, 1, 2, \cdots, n. \tag{3}$$

Differentiating equation (2) n times gives

$$R^{[n]}(x) = f^{[n]}(x) - P^{[n]}(x).$$

But since $P(x)$ is a polynomial of degree n, its nth derivative is a constant,

$$P^{[n]}(x) = P^{[n]}(a) = f^{[n]}(a), \quad \text{and}$$
$$R^{[n]}(x) = f^{[n]}(x) - f^{[n]}(a).$$

By the law of the mean

$$R^{[n]}(x) = f^{[n+1]}(\theta)(x-a), \qquad \theta \text{ between } a \text{ and } x. \tag{4}$$

Let M and m be respectively the maximum and minimum values of $f^{[n+1]}(\theta)$ in the interval (a, x). Then if $a \leqq t \leqq x$,

$$m(t-a) \leqq R^{[n]}(t) \leqq M(t-a).$$

It is clear geometrically that if an ordinate of one curve is never greater than the corresponding ordinate of a second curve in a given interval, then the area under the first curve cannot exceed the area under the second for this interval. Accordingly, we integrate this inequality from a to x and obtain

$$\tfrac{1}{2}m(x-a)^2 \leqq R^{[n-1]}(x) \leqq \tfrac{1}{2}M(x-a)^2.$$

Continuity of the $(n+1)$st derivative of $f(t)$ now implies

$$R^{[n-1]}(x) = f^{[n+1]}(\theta_1)\frac{(x-a)^2}{2}, \qquad \theta_1 \text{ between } a \text{ and } x.^* \tag{5}$$

If $x \leqq t \leqq a$, the inequalities are reversed and (5) is again true.

Repeated integration of the inequalities from a to x will produce

$$R(x) = \frac{f^{[n+1]}(\phi)}{(n+1)!}(x-a)^{n+1}, \qquad \phi \text{ between } a \text{ and } x.$$

Equation (2) now gives

$$f(x) = P(x) + R(x) \tag{6}$$

which is Taylor's Theorem with remainder.

It may be noted that the question of convergence has been completely sidestepped. $P(x)$ is a good approximation of $f(x)$ whenever the least upper bound for $|R(x)|$ is as small as desired. When n is finite, equation (6) is identically true in x, and the question of the series representing the function does not arise.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 896. *Proposed by F. J. Duarte, Caracas, Venezuela*

If p is an odd prime show that

$$[1\cdot 3\cdot 5\cdots(p-2)]^2 \equiv (-1)^{(p+1)/2} \pmod{p}.$$

E 897. *Proposed by W. R. Ransom, Tufts College*

A cube is divided into equal parts by the plane of a hexagon whose vertices are midpoints of its edges. In what ratio is a cube divided by a parallel plane that divides an edge in the ratio $a:(1-a)$?

E 898. *Proposed by N. S. Mendelsohn, University of Manitoba*

A pack of N cards is disposed of as follows. The top card is placed at the bottom of the deck, the next card is discarded, the third card is placed at the bottom, the fourth discarded, *etc.*, this process being carried on until there is only

* This follows more directly and more rigorously from equation (4) by the first law of the mean for integrals if this is available.

one card left. Let $f(N)$ be the position of this card, from the top, in the original arrangement of the deck. Prove that

$$f(N) = 2N - 2^{[\log_2 N]+1} + 1,$$

where $[x]$ denotes, as usual, the largest integer not greater than x.

E 899. *Proposed by E. R. Bowersox, Chicago, Illinois*

Construct an equilateral triangle given the distances of a point in its plane from the three vertices.

E 900. *Proposed by N. A. Court, University of Oklahoma*

If four spheres may be divided into two pairs so that the circle common to one pair is cospherical with the circle common to the second pair, then the circle determined by any two of the given spheres is cospherical with the circle determined by the remaining two spheres.

E 849 [1949, 552]. Correction

In both the title and the solution of this problem read " pedal" for "cevian."

SOLUTIONS

A Curious Representation of Integers

E 861 [1949, 262]. *Proposed by Vern Hoggatt, College of Puget Sound, and Leo Moser, University of Manitoba*

Let a be any positive number different from 1 and let p be any integer greater than 3. Show that every integer may be expressed by using p a's and a finite number of operator symbols used in high school texts.

Solution by the Proposers. It is easily verified that

$$\pm \log_{\log \sqrt{a}\, a} \log \sqrt{\sqrt{\cdots \sqrt{a}}}\, a = \pm n, \tag{1}$$

$$\pm \log_{(a+a)/a} \log \sqrt{\sqrt{\cdots \sqrt{a}}}\, a = \pm n, \tag{2}$$

$$(a - a) + (a - a) = (a - a) + \log_a \log_a a = 0, \tag{3}$$

where, in (1) and (2), there are n radical signs in the iterated set. These formulas solve the problem for $p=4$ and 5. If $p>5$, then add to the appropriate left member a suitable number of terms of the form $(a-a)$.

Hoggatt has also shown that every integer can be expressed by using the first $m>3$ positive integers, repetitions allowed, and a finite number of operator symbols used in high school texts.

Extreme Rectangles About a Parallelogram

E 862 [1949, 262]. *Proposed by R. E. Horton, Los Angeles City College*

Find the rectangles of greatest and least area which can be circumscribed about a given parallelogram.

Solution by Roger Lessard, École Polytechnique, Montreal. Let a and b, $a \geqq b$, be the sides of the parallelogram P, and $\alpha \leqq \pi/2$ the angle between them. Let β be the acute angle made with a by a side of the rectangle passing through the vertex of a non-acute angle of P. Then we must have $0 \leqq \beta \leqq \alpha$. Denote the areas of the rectangle and the parallelogram by A_R and A_P respectively. Then

$$\begin{aligned} A(\beta) &\equiv A_R - A_P \\ &= a^2 \sin\beta \cos\beta + b^2 \sin(\alpha - \beta)\cos(\alpha - \beta) \\ &= (1/2)a^2 \sin 2\beta + (1/2)b^2 \sin 2(\alpha - \beta), \end{aligned} \tag{1}$$

whence

$$dA/d\beta = a^2 \cos 2\beta - b^2 \cos 2(\alpha - \beta) \tag{2}$$

and

$$d^2A/d\beta^2 = -2a^2 \sin 2\beta - 2b^2 \sin 2(\alpha - \beta) = -4A < 0.$$

Since there is no turning point minimum, the least area will be at one end of the interval $(0, \alpha)$. Now

$$A(0) = (1/2)b^2 \sin 2\alpha \leqq (1/2)a^2 \sin 2\alpha = A(\alpha).$$

Therefore

$$\min A_R = (a + b\cos\alpha)b \sin\alpha.$$

Setting $dA/d\beta = 0$ we find

$$\tan(2\beta - \alpha) = [(a^2 - b^2)/(a^2 + b^2)] \cot\alpha.$$

Therefore, since $2\beta - \alpha \leqq \alpha$, we have a turning point maximum only if

$$(a^2 - b^2)/(a^2 + b^2) \leqq \tan^2\alpha, \tag{3}$$

in which case we find

$$\max A_R = ab \sin\alpha + [a^4 - 2a^2b^2 \cos 2\alpha + b^4]^{1/2}/2.$$

If (3) is not satisfied then (2) is positive and (1) is continuously increasing in the interval $(0, \alpha)$, and the greatest area is obtained at $\beta = \alpha$. Thus, in this case, we have

$$\max A_R = (b + a\cos\alpha)a \sin\alpha.$$

Also solved by D. H. Browne, Sam Kravitz, N. D. Lane, C. S. Ogilvy, I. W. Thompson, and the proposer.

Let the parallelogram be $ABCD$, $AB \leqq BC$, and the circumscribed rectangle $PQRS$, such that A lies on PQ, B on QR, C on RS, S on SP. Let O be the common center of the parallelogram and rectangle. Browne showed that a turning point maximum exists if $\sphericalangle ABO - \sphericalangle OAB < \pi/2$, and that in this case $\sphericalangle OBR = \sphericalangle RCO$. A euclidean construction of this maximum rectangle can be made, and

we have

$$\max A_R = (AC)(BD)\sin^2(135^\circ - BOC/2).$$

The Generalized Steiner-Lehmus Problem

E 863 [1949, 263]. *Proposed by W. O. Pennell, Exeter, N. H.*

If $\alpha>0$, $\beta>0$, $\alpha+\beta<\pi$, $0<k<1$, then $\alpha=\beta$ is a necessary and sufficient condition for

$$\sin\alpha\sin(k\alpha+\beta) = \sin\beta\sin(k\beta+\alpha).$$

I. *Solution by Mary Payne, Michigan State College.* If $\alpha=\beta$, the identity is obviously true without any restrictions on k or on the size of α and β.

If the identity is true, we may differentiate it twice with respect to k to obtain

$$-\alpha^2\sin\alpha\sin(k\alpha+\beta) = -\beta^2\sin\beta\sin(k\beta+\alpha).$$

Dividing this result by the identity, we have

$$\beta^2 = \alpha^2 \quad \text{or} \quad \beta = \pm\alpha,$$

and if $\alpha>0$, $\beta>\alpha$, the truth of the identity implies $\alpha=\beta$ with no restrictions on k and no further restrictions on α and β.

Since the identity is evidently true for $\alpha=\pm\beta$, we may state the following more general theorem: A necessary and sufficient condition that the identity

$$\sin\alpha\sin(k\alpha+\beta) = \sin\beta\sin(k\beta+\alpha)$$

hold is that $\alpha=\pm\beta$.

II. *Solution by N. D. Lane, St. Andrew's College, Aurora, Ontario.* Evidently, if $\alpha=\beta$, the equation is satisfied. Suppose that $\alpha>\beta$. Multiplying by 2 we get

$$-\cos(k\alpha+\beta+\alpha)+\cos(k\alpha-\alpha+\beta) = -\cos(k\beta+\alpha+\beta)+\cos(k\beta-\beta+\alpha),$$

or

$$\cos(k\beta+\alpha+\beta)-\cos(k\alpha+\beta+\alpha) = \cos(k\beta-\beta+\alpha)-\cos(k\alpha-\alpha+\beta),$$

whence

$$\sin[(k+2)(\beta+\alpha)/2]\sin[k(\beta-\alpha)/2] = \sin[k(\beta+\alpha)/2]\sin[(2-k)(\alpha-\beta)/2].$$

Now if $\alpha>\beta$, the left side is negative, while the right side is positive, and we have a contradiction.

Also solved by B. B. Dressler, L. M. Kelly, Norman Miller, C. S. Ogilvy, and the proposer.

Editorial Note. Consider a triangle ABC with base angles at A and B denoted by α and β respectively. Let AD and BE be drawn so that $\sphericalangle BAD = k\alpha$ and $\sphericalangle ABE = k\beta$. Then, by the law of sines applied to triangles ABE and ABD,

$$\sin \alpha / \sin (k\beta + \alpha) = BE/AB,$$

$$\sin \beta / \sin (k\alpha + \beta) = AD/AB.$$

Therefore, if $BE = AD$,

$$\sin \alpha \sin (k\alpha + \beta) = \sin \beta \sin (k\beta + \alpha),$$

and, by the proposed problem, $\alpha = \beta$, and triangle ABC is isosceles. If $k = 1/2$, then AD and BE are the angle bisectors of α and β and we have the Steiner-Lehmus theorem: "If the bisectors of the base angles of a triangle are equal, then the triangle is isosceles."

A Well Known Identity

E864 [1949, 263]. *Proposed by N. S. Mendelsohn, University of Manitoba*

Prove that

$$\sum_{n=1}^{r} 1/n = \sum_{n=1}^{r} (-1)^{n+1}(1/n)\binom{r}{n}.$$

Solution by Leo Moser, University of Manitoba. Evaluate

$$\int_0^1 \frac{1 - x^r}{1 - x}\, dx$$

directly to obtain the left side of the given expression. Then evaluate the same integral after making the substitution $x = 1 - u$ to obtain the right side of the given expression.

Also solved by W. D. Berg, W. G. Brady, D. H. Browne, Paul Carnahan, Ragnar Dybvik, B. F. Hadnot, R. Hamming, Vern Hoggatt, M. S. Klamkin, Roger Lessard, Norman Miller, S. T. Parker, C. F. Pinzka, C. M. Sandwick, C. W. Trigg, and the proposer.

Editorial Note. Many references were sent in for this problem. See, for example, prob. 38, p. 6, Pólya and Szegö, *Aufgaben und Lehrsätze aus der Analysis*; prob. 21, p. 393, Hardy, *Pure Mathematics*; prob. 23, p. 169, Hall and Knight, *Higher Algebra*; p. 327, Jordan, *Calculus of Finite Differences*; prob. 18, v. 2, p. 19, Chrystal, *Algebra*; prob. 117(a), p. 270, Knopp, *Infinite Series*; p. 104, ser. 2, v. 1, 1891, *Mathesis*; prob. 1245, Jan. 1933, *School Science and Mathematics*; problems 4130 [1945, 527] and E 460 [1941, 700], this MONTHLY. For other problems in the MONTHLY dealing with sums involving the binomial coefficients see E 205, E 445, E 670, 3625, 3701, 3748, and for other problems dealing with the sum of a finite number of terms of the harmonic series see E 46, E 53, E 520, E 819, 3652, 4267.

Equivalent Concurrent Sections of a Tetrahedron

E 865 [1949, 263]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find a point such that planes drawn through this point parallel to the faces of a tetrahedron cut the opposite trihedrals in equivalent triangles. Express the common area of these triangles in terms of the areas of the faces of the tetrahedron.

Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn. Denote the areas of the faces of the tetrahedron by A_i $(i=1, 2, 3, 4)$, the corresponding altitudes by H_i, and the perpendicular distances from the corresponding vertices to the corresponding parallel sections by h_i. Let V be the volume of the tetrahedron and A the common area of the parallel sections. Then

$$\Sigma(H_i - h_i)A_i = 3V = H_iA_i, \qquad h_i^2/H_i^2 = A/A_i.$$

Therefore

$$3V = \Sigma\left\{H_iA_i - H_i(AA_i)^{1/2}\right\} = 12V - \Sigma H_i(AA_i)^{1/2},$$

or

$$A^{1/2} = 9V/\Sigma H_iA_i^{1/2} = 3/\Sigma A_i^{-1/2}.$$

Also

$$h_i = 3H_iA_i^{-1/2}/\Sigma A_i^{-1/2}.$$

Also solved by N. D. Lane and the proposer.

The analogous problem for the plane was proposed by J. Neuberg as question 30, p. 148, 1881, *Mathesis*. For this case we have

$$2/k = 1/a + 1/b + 1/c,$$

where a, b, c are the sides of the triangle and k is the common length of the concurrent lines which are drawn parallel to the sides of the triangle. This, and the result of the given problem, suggested to Lane for the corresponding problem of an n-dimensional simplex the formula

$$n/k = \Sigma 1/a,$$

where k^{n-1} is the common content of the $(n-1)$-dimensional cells parallel to the $(n-1)$-dimensional cell "faces" of the simplex, and a^{n-1} is the content of a "face."

A Lattice Game

E 866 [1949, 338]. *Proposed by L. J. Burton, Bryn Mawr College*

Players A and B take turns, beginning with A, each marking a previously unmarked unit line segment joining two points with integral coordinates at a unit distance in a plane.

(a) Prove that B can prevent A from ever marking all the line segments in the perimeter of any closed polygon.

(b) Prove that if $P_1:(x_1, y_1)$ and $P_2:(x_2, y_2)$ are any two fixed points with integral coordinates such that $|P_1P_2| > 1$, then B can prevent A from joining P_1 and P_2 by a broken line consisting of segments marked by A.

Solution by the Proposer. (a) B simply prevents A from ever making an L with two of his segments. Whenever A draws a vertical segment, B completes the L by drawing the proper horizontal segment, and whenever A draws a horizontal segment, B completes the L by drawing the proper vertical segment. Of course every closed polygon made of unit segments joining points with integral coordinates has at least one L.

(b) Draw the line P_1P_2. By interchanging the axes if necessary, we may assume that P_1P_2 is not vertical. Whenever A marks a segment lying along P_1P_2 (if this is possible), B marks another segment lying along the line, which will be possible since $|P_1P_2| > 1$. When A draws a segment with at least one vertex above P_1P_2, B completes the inverted backward L at a vertex above P_1P_2 if the slope of $P_1P_2 \geqq 0$, and completes the inverted L at a vertex above P_1P_2 if the slope < 0. If no vertex of A's segment is above P_1P_2 and if at least one vertex is below P_1P_2, B completes the backward L at a vertex below P_1P_2 if the slope $\geqq 0$, and completes the L at a vertex below P_1P_2 if the slope < 0.

It is clear that if A leaves P_1P_2 he must use an angle of the kind drawn to reach P_2, but B prevents him from making such an angle. It is fairly evident that the case when A's segment crosses P_1P_2 causes no difficulty, because if A ever gets above P_1P_2 on his route, he can never get down to the line again.

A Limit Given by John Wallis

E 867 [1949, 338]. *Proposed by Walter Fleming, Fort Hays Kansas State College*

Find

$$\lim_{n\to\infty} n^{-p} \sum_{j=1}^{n} j^{p-1}.$$

Solution by S. T. Thompson, Tacoma, Washington. The definition of a Riemann integral shows that the limit is $1/p$ if $p \geqq 1$. If $0 < p < 1$ the corresponding integral is improper but converges to $1/p$, and the limit is still $1/p$. The familiar p-series of elementary calculus shows that the limit is $+\infty$ if $p \leqq 0$.

Also solved by Philip Anselone, W. D. Berg, Louis Berkofsky, A. B. Boggs, D. H. Browne, Paul Carnahan, K. L. Cooke, B. B. Dressler, Maurice Dunn, Vern Hoggatt, Philip Kirmser, N. D. Lane, Roger Lessard, A. E. Livingston, Joseph Rosenbaum, and the proposer. Many of these solutions were incomplete.

As Azriel Rosenfeld pointed out, this limit was established by John Wallis in his *Opera Mathematica* in 1656. See, *e.g.*, *Scripta Mathematica*, vol. IX, no. 4, p. 241. Boggs located the problem in Pólya and Szegö, *Aufgaben und Lehrsätze aus der Analysis*, probs. 20 and 22, p. 39, vol. 1.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results found in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4375. *Proposed by N. J. Fine, University of Pennsylvania*

Let $((x))=x-[x]-\frac{1}{2}$. Prove that the sums

$$\sum_{n=1}^{m}((2^n x+\tfrac{1}{2}))$$

are uniformly bounded.

4376. *Proposed by Victor Thébault, Tennie, Sarthe, France*

For a tetrahedron $ABCD$ let (A) be the sphere with center A and radius equal to the altitude AA', and let the tangent planes to (A) through the edges BC, CD, DB intersect in the point A_1. Let B_1, C_1, D_1 be analogously defined. If $ABCD$ is orthocentric show that AA_1, BB_1, CC_1, DD_1 are concurrent at the isogonal conjugate of the orthocenter, and that otherwise AA_1, BB_1, CC_1, DD_1 constitute a hyperbolic group of lines.

4377. *Proposed by Paul Brock, Reeves Instrument Company, New York City*

Consider two non-decreasing sequences of n positive integers, $(n<M)$, the integers being chosen at random. What is the most probable number of integers in the sequence containing the maximum integer, each of which is larger than the maximum of the second sequence?

4378. *Proposed by H. W. Smith, Oklahoma Agricultural and Mechanical College.*

Determine

$$\lim_{n\to\infty}\frac{\sum_{k=0}^{n}{}_nC_k/(2k+1)}{\sum_{k=0}^{n+1}{}_{n+1}C_k/(2k+1)},$$

where the ${}_nC_k$ are the binomial coefficients.

4379. *Proposed by Watson Fulks, University of Minnesota*

If

(1) $\phi(t)$ is defined in I: $m \leqq t \leqq M$,

(2) for any a, b in I, $a \neq b$,

$$\phi\left(\frac{a+b}{2}\right) < \frac{\phi(a)+\phi(b)}{2},$$

(3) p_i are non-negative numbers, $i = 1, 2, 3, \cdots, n$,

(4) $t_i \in I$, not all t_i equal,

it has been shown that

$$\phi\left(\frac{\sum_{i=1}^{n} p_i t_i}{\sum_{i=1}^{n} p_i}\right) \leqq \frac{\sum_{i=1}^{n} p_i \phi(t_i)}{\sum_{i=1}^{n} p_i}.$$

See Pólya and Szegö, *Aufgaben und Lehrsätze aus der Analysis*, v. I, p. 53, problem 74.

Show that the relation is true without the equality sign.

SOLUTIONS

Orthocentric Tetrahedron, Minimum Property

4251 [1947, 286]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In an orthocentric tetrahedron the lines joining the symmedian points (Lemoine points) of the faces to the midpoints of the corresponding altitudes, are concurrent at a point such that the sum of the squares of its distances to the planes of the faces is a minimum.

Note. The problem is solved by the following result due to Robert Bouvaist which appeared in *Mathesis*, t. LVI, 1947, p. 113.*

Let (Γ) be a quadric inscribed in the tetrahedron $ABCD$; let its points of contact with the faces BCD, CDA, DAB, ABC be A', B', C', D'. We restrict our attention to the case for which AA', BB', CC', DD' are concurrent at P. If P varies on AA' the one-parameter family of quadrics (Γ) includes the points A and A' and the conic (γ_1) inscribed in BCD at the feet of the cevians of A'. Hence the locus of the centers of the pencil of quadrics is the line M_1O_1 joining the midpoint M_1 of AA' to the center O_1 of (γ_1). Thus the four lines M_1O_1, M_2O_2, M_3O_3, M_4O_4 pass through the center of (Γ).

If $ABCD$ is orthocentric the point P is the orthocenter, and we have the following proposition: In an orthocentric tetrahedron the lines joining the symmedian points (Lemoine points) of each face to the midpoint of the correspond-

* Translation by W. E. Byrne, Virginia Military Institute.

ing altitude pass through the first Lemoine point of the tetrahedron.[1]

It is known that in a triangle the conic tangent to the three sides at the feet of the altitudes has as its center the Lemoine point of the triangle. Likewise in an orthocentric tetrahedron the quadric tangent to the faces at the feet of the altitudes has as its center the first Lemoine point of the tetrahedron.[2]

The Four Liars

4288 [1948, 165]. *Proposed by J. W. Campbell, University of Alberta.*

In his Messenger Lecture on Probability (A. S. Eddington, *New Pathways in Science*, 1935, p. 121) Eddington referred to the following problem: If A, B, C, D each speak the truth once in three times (independently), and A affirms that B denies that C declares that D is a liar, what is the probability that D was telling the truth? He used the Exclusion Method of Solution and arrived at the numerical result 25/71.

Prove that the correct probability is 13/41, and that this is also the probability that each of A, B, and C told the truth.

I. *Solution by Bart Park, Michigan College of Mining and Technology.* Of the 81 possible combinations of truth and lie, the only ones which are compatible with the statement, "A affirms that B denies that C declares that D is a liar," are those which contain an even number of lies. Thus, if D tells the truth, all the others must tell the truth or exactly two of them must lie. These combinations or paths may be represented as follows:

T-T-T-T,	T-L-L-T,	L-T-L-T,	L-L-T-T.
(1 way)	(4 ways)	(4 ways)	(4 ways)

Similarly, if D tells a lie the following paths are possible:

L-L-L-L,	T-T-L-L,	L-T-T-L,	T-L-T-L.
(16 ways)	(4 ways)	(4 ways)	(4 ways)

Therefore the probability that D was telling the truth is 13/41. Exactly similar reasoning applies to A, B and C.

II. *Solution by G. N. Garrison and Theodore Hailperin, Lehigh University.* The following notations will be used:

α	for	A speaks the truth about B's statement
β	for	B speaks the truth about C's statement
γ	for	C speaks the truth about D's statement
δ	for	D speaks the truth about an event X
$\overline{E}$	for	the contrary event to E
EF	for	the event "E and F"
$E \vee F$	for	the event "E or F"

[1] The first Lemoine point of a tetrahedron is a point such that the sum of the squares of its distances to the faces of the tetrahedron is a minimum.

[2] R. Bouvaist et V. Thébault, Comptes-Rendus de l'Académie des Sciences, Paris, 1943, pp. 418–419.

The probabilities of $\alpha, \beta, \gamma, \delta$ are denoted by a, b, c, d; and $\bar{x}$ is used for $1-x$.

The event "C declares that D lied" may be expressed as

$$\gamma\bar{\delta} \vee \bar{\gamma}\delta,$$

for the event will happen if D lies and C speaks the truth about D's statement, or if D speaks the truth and C lies. The event "B denies that C declares that D lied" then is

$$\beta\overline{(\gamma\bar{\delta} \vee \bar{\gamma}\delta)} \vee \bar{\beta}(\gamma\bar{\delta} \vee \bar{\gamma}\delta)$$

or, by the calculus of logic,

$$\beta\gamma\delta \vee \beta\bar{\gamma}\bar{\delta} \vee \bar{\beta}\gamma\bar{\delta} \vee \bar{\beta}\bar{\gamma}\delta.$$

Finally "A affirms that B denies that C declares that D lied" becomes

$$\alpha(\beta\gamma\delta \vee \beta\bar{\gamma}\bar{\delta} \vee \bar{\beta}\gamma\bar{\delta} \vee \bar{\beta}\bar{\gamma}\delta) \vee \bar{\alpha}\overline{(\beta\gamma\delta \vee \beta\bar{\gamma}\bar{\delta} \vee \bar{\beta}\gamma\bar{\delta} \vee \bar{\beta}\bar{\gamma}\delta)}$$

which is

$$\alpha\beta\gamma\delta \vee \alpha\beta\bar{\gamma}\bar{\delta} \vee \alpha\bar{\beta}\gamma\bar{\delta} \vee \alpha\bar{\beta}\bar{\gamma}\delta \vee \bar{\alpha}\bar{\beta}\gamma\delta \vee \bar{\alpha}\beta\bar{\gamma}\delta \vee \bar{\alpha}\beta\gamma\bar{\delta} \vee \bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}.$$

We note that this expression is invariant under the cyclical substitution $(\alpha\beta\gamma\delta)$, since the first and last members of the alternation are invariant and the remaining six include all of the possible cases where two of the events are denied; we also note that these alternatives are mutually exclusive and consist of the conjunction of independent events. Thus the probability of the event is

$$abcd + ab\bar{c}\bar{d} + a\bar{b}c\bar{d} + a\bar{b}\bar{c}d + \bar{a}\bar{b}cd + \bar{a}\bar{b}\bar{c}d + \bar{a}\bar{b}c\bar{d} + \bar{a}\bar{b}\bar{c}\bar{d},$$

and the probability that this event and δ happened is obtained by deleting those terms containing $\bar{d}$. Thus the *à posteriori* probability of δ on the hypothesis that "A affirms that $B \cdots$" is

$$\frac{abcd + a\bar{b}\bar{c}d + \bar{a}\bar{b}cd + \bar{a}\bar{b}\bar{c}d}{abcd + ab\bar{c}\bar{d} + a\bar{b}c\bar{d} + a\bar{b}\bar{c}d + \bar{a}\bar{b}cd + \bar{a}\bar{b}\bar{c}d + \bar{a}\bar{b}c\bar{d} + \bar{a}\bar{b}\bar{c}\bar{d}}.$$

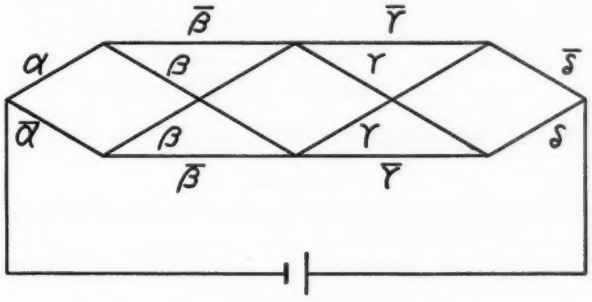
If we put $a=b=c=d=1/3$, we obtain 13/41. Clearly this is also the *à posteriori* probability of α, β, and γ because of the previously mentioned symmetry.

Also solved by Karl Itkin, John Riordan, and the Proposer.

Editorial Note. Riordan cites the following problem as one to which the same analysis applies. Let α, β, γ, and δ be interpreted as switching variables (C. E. Shannon, *A Symbolic Analysis of Relay and Switching Circuits*, Trans. A.I.E.E., 1938), where if γ is a switch, then $\bar{\gamma}$ is its negative, closed when γ is open and vice versa. Given that the circuit shown below is open and that each switch operates with probability 1/3, what is the probability that a specified switch is operated?

All the solvers made the assumption that $\bar{\gamma}$, for example, means "C asserts the opposite of D's statement." Without such assumptions about B and C we

are not sure that either of them made any statement whatever—perhaps A fabricated the whole chain. For discussion and solution of this more general interpretation see (references supplied by C. D. Olds) *Math. Gazette* 1935, pp. 256–257; 1936, pp. 298–308, 309–310. In the last reference, the interpretation is given which justifies Eddington's original result 25/71.



Twisted Curves

4292 [1948, 253]. *Proposed by R. Goormaghtigh, Bruges, Belgium*

If s and ρ are the arc length and the radius of curvature of a plane curve Γ at a variable point M, and if it be required that s have a constant ratio to the distance of M from a fixed point, then Γ must be a cycloidal curve and

$$\lambda^2 s^2 - \rho^2 = a^2,$$

λ and a being constants.

Prove that, in the case of a twisted curve, the condition is

$$\lambda^2 s^2 - \rho^2 = \left(\int \frac{\rho}{\tau} ds + a \right)^2,$$

τ being the radius of torsion of Γ at M.

Solution by the Proposer. Consider variable axes of coördinates, these being the tangent Mx, the binormal My and the principal normal Mz of Γ at M. Then, α, β, γ being the coördinates of a fixed point,

$$(1) \qquad d\alpha/ds = \gamma/\rho - 1, \quad d\beta/ds = \gamma/\tau, \quad d\gamma/ds = -\alpha/\rho - \beta/\tau,$$

(see Cesàro-Kowalewski, *Natürliche Geometrie*, Teubner, 1901, p. 157, also Lane, *Metric Differential Geometry of Curves and Surfaces*, p. 47, ex. 3) and

$$(2) \qquad \alpha^2 + \beta^2 + \gamma^2 = k^2 s^2.$$

Differentiating (2), taking (1) into account,

$$\alpha = -k^2 s;$$

differentiating again,

$$\gamma = (1 - k^2)\rho,$$

and again

$$\beta = \tau\{k^2 s/\rho - (1 - k^2)d\rho/ds\}.$$

Substitution of these values of α, β, γ into (2) gives a relation which can be put in the form

$$\frac{\rho}{\tau} = \frac{k^2 s/(1 - k^2) - \rho d\rho/ds}{\{k^2 s^2/(1 - k^2) - \rho^2\}^{1/2}}$$

whence the theorem follows immediately, $\lambda^2 = k^2/(1-k^2)$, $-1 < k < 1$.

Summation

4293 [1948, 254]. *Proposed by H. F. Sandham, Trinity College, Ireland*

Evaluate

$$\sum_{n=1}^{\infty} \frac{\left(\frac{3 - \sqrt{5}}{2}\right)^n}{n^3}.$$

Solution by D. A. Darling, Rutgers University. The result is

$$\frac{2}{15}\left\{6\zeta(3) + \pi^2 \log\left(\frac{-1 + \sqrt{5}}{2}\right) - 5\left[\log\left(\frac{-1 + \sqrt{5}}{2}\right)\right]^3\right\},$$

where $\zeta(s)$ is the Riemann zeta function. This is an immediate consequence of a formula due to Spence to the effect that if

$$\phi(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^3}, \qquad |x| \leqq \tfrac{1}{2},$$

then

$$\phi\left(\frac{x}{x-1}\right) + \phi(x) + \phi(1 - x) - \phi(1)$$

$$= \frac{\pi^2}{6} \log(1 - x) + \frac{1}{6}\{\log(1 - x)\}^2\{\log(1 - x) - 3 \log x\}.$$

See W. Spence, *An essay on the theory of the various orders of logarithmic transcendents*, (1809), p. 28. The formula was discovered independently by Ramanujan. In the *Journal of the London Mathematical Society*, v. 3 (1928), p. 217, G. N. Watson gives an elementary proof based upon the obvious relation

$$\phi(x) = \frac{x}{2}\int_0^1 \frac{\log(1 - u)^2 du}{1 - x + xu},$$

expressing the left-hand side of Spence's formulas as the sum of four such integrals and making elementary transformations.

Letting $x=(3-\sqrt{5})/2$ we see that $(x-1)^2=x$ and $x/(x-1)=x-1$, and putting further $v=1-x$, Spence's formula gives

$$\phi(x)+\phi(v)+\phi(-v)=\phi(1)+\frac{\pi^2}{6}\log v-\frac{5}{6}(\log v)^2.$$

From the series definition it is easy to see that

$$\phi(v)+\phi(-v)=\frac{2}{2^2}\phi(v^2)=\frac{1}{4}\phi(x),$$

and observing finally that $v=(-1+\sqrt{5})/2$, $\phi(1)=\zeta(3)$, we obtain the result announced.

Also solved by the Proposer.

The Identity as a Product of Successive Elements

4300 [1948, 369]. *Proposed by Leo Moser, University of Toronto*

Let $a_1, a_2, \cdots, a_n$ be n, not necessarily distinct, elements of a group of order n. Show that there exist integers p and q, $1\leqq p\leqq q\leqq n$, such that

$$\prod_{i=p}^{q} a_i = 1.$$

Solution by D. J. Newman, Freshman, College of the City of New York. Consider $1, a_1, a_1a_2, a_1a_2a_3, \cdots, a_1a_2a_3\cdots a_n$. Since there are $n+1$ elements of the group present in this set, we must have at least two of them equal. If $1=\prod_{i=1}^{k}a_i$, then the proposed theorem is proved. Otherwise

$$\prod_{i=1}^{q} a_i = \prod_{i=1}^{v} a_i, \qquad q>v,$$

whence

$$\prod_{i=v+1}^{q} a_i = 1.$$

With $p=v+1$, this completes the proof.

Also solved by Michael Aissen, R. V. Andree, Paul Brock, S. H. Gould, B. A. Hausmann, J. B. Kelly, Arnold Walfisz, and the Proposer. Kelly notes the corollary: From any row of n integers, a block of adjacent integers may always be selected whose sum is divisible by n.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

Applied Differential Equations. By F. E. Relton. Blackie and Son, Ltd. Glasgow, 1948. 4+264 pages. 20s.

This book is divided into eleven chapters with the following headings. I. Preliminary notions; hyperbolic functions. II. First order; standard methods. III. The linear equations; constant coefficients. IV. Miscellaneous theorems and methods. V. Simultaneous equations. VI. Fourier series. VII. Partial differential equations. VIII. The method of isoclinals. IX. Numerical methods of solution. X. Equations in three variables. XI. Variable coefficients.

As these headings indicate the contents of the book is essentially that of any good American introductory textbook on differential equations. The arrangement of its material is, however, somewhat different.

The book was written primarily for students of applied science. This is made evident by the numerous applications which are developed in detail throughout the text or are listed among the exercises (given with hints and answers). These include problems in mechanical vibrations, heat flow, and bending of beams. It is also made evident by the emphasis upon the intuitive or formal treatment of the various topics. As, for example in the chapters on Fourier series and on numerical methods, no conditions are stated for the convergence of the series or of the other processes.

Throughout the book the author makes statements which attempt to justify this omission of theorems and proofs. For example, in the preface, he says: " 'Epsilonology' is to the mathematician what J. S. Bach is to the musician; there is an enormous sense of gratification in its essential rightness. It is an exhilarating experience to go right down to the foundations of one's subject; especially if one does not stay there. . . . But I see no justification on that account for expecting those to whose studies mathematics is ancillary to master the philosophical profundities of a limit. It would be as rational to expect the piano-tuner to be interested in the comma of Pythagoras. Geometric intuition is good enough for the applied scientist."

But *is* geometric intuition good enough for the applied scientist? With the increasing degree of accuracy required in modern manufacturing processes, the engineer is (or should be) more conscious than ever of the importance of questions of convergence. Many have learned from experience that it is foolhardy to be ignorant of the restrictions attached to the use of a mathematical process; that is, they realize the importance of knowing what hypotheses are needed to ensure certain conclusions. While it may be conceded that an epsilon-delta proof does not often appeal to an engineer, he nevertheless wishes to know at least the

gist of a proof so that he might understand the reasons for the validity of the theorem. In the experience of the reviewer, neither graduate nor undergraduate engineering students regard a discussion of theorems and proofs as worthless or uninteresting.

In spite of the above criticisms, the reviewer considers the book to be a clear and well-organized exposition of the subject. The reader will particularly enjoy the author's numerous informal remarks, both for their content and language. The book would make good collateral reading in an introductory course on differential equations or, if somewhat supplemented with respect to theorems and proofs, would also be adaptable as a textbook for such a course.

MORRIS MARDEN

Theory of Experimental Inference. By C. W. Churchman. New York, The Macmillan Company, 1948. 11+292 pp. $4.25.

This book attempts "to preach the gospel of modernism to two groups"—the philosophers and the scientists. To the modern philosopher of science, it preaches that "we can no longer conceive of scientific method in the older empirical fashion of 'discrete' observation sentences"; that "we put far more into the process of question-answering than is contained in so much of present-day philosophical writing"; and that "the simplest question of fact in science requires for even an approximation, a judgment of value." (p. vii) To the scientist, the book has tried "to preach the moral that the process of raising a question and proceeding to reply to it, is a process which eventually demands the active cooperation of all fields of research." Moreover, the author has attempted "to show that besides certain statistical considerations, the self conscious experimenter must take into account very general problems concerning the natural universe within which he is solving his special problems." (p. viii) However praiseworthy his endeavor, Churchman has substantially failed in giving clear statement to his brand of "modernism," and his lack of clarity suggests fundamental weaknesses, inaccuracies, and confusions.

The first three chapters are concerned with the nature of statistical tests, the general methodology of inference and certain problems of method. The next six chapters form a backdrop for the author's presentation of his own position, called "experimentalism." One by one, traditional and current positions fall under Churchman's attack. It turns out that, for the author, the fundamental problems of the philosophy of science are: "What can be known?" and "How can it be known?" Or, more specifically, the important problem is how to define "answer" in such a way that answers exist even though (1) "the answering of any question of law presupposes the answering of at least some questions of fact" and (2) "the answering of any questions of fact presupposes the answering of at least some questions of law." (pp. 47, 51, 57)

The remainder of the work is devoted to the definition of *experimentalism* and its applications. Experimentalism is characterized by its fundamental postulate, namely, that "*there exists a formalization of nature, such that stochastic limits exist*

for certain sequences of mathematical functions of the observations which are pertinent to a given question of fact." (p. 178) One wonders whether the complexity of the machinery developed under the name of experimentalism, is warranted by the questions Churchman seeks to answer. The essay is concluded with discussions of chance, loss, risk, and quality control. It is the appeal of the essay "that the sciences, and society in general, collaborate in setting up a controlled science of losses and risks, i.e., a controlled science of ethics." (p. 263)

To the professional philosopher, further objections occur. Churchman seems to have confused methodology of science with epistemology, and (at times) both of these fields with metaphysics. Another serious confusion is the author's repeated claim that science requires ethics and that ethics can be a science. Perhaps the only way to understand his claim is to realize that, throughout, a form of utilitarianism has been tacitly assumed, but surely not scientifically established.

V. G. HINSHAW, JR.

College Algebra by E. A. Cameron and E. T. Browne. New York, Henry Holt and Company, 1949. 10+406 pp. $3.00.

The topics covered in this book are essentially those contained in the standard college algebra text, the first nine of twenty-four chapters containing a thorough review of elementary algebra. The authors have been generally successful in their aim to treat this review material from a more mature point of view than is possible in a high school course. Although some attempt has been made to show that the formal rules of manipulation of algebraic quantities are consequences of the postulates assumed for the number systems, in most instances the common practice of stating rules of procedure is adopted.

The theorems are carefully stated, often in more general form than is customary, and the proofs are concise and accurate.

The theory of equations, including the quadratic equation, is especially well done. Descartes' Rule of Signs and the theorem on rational roots are examples of theorems stated and proved in their general forms. Only second and third order determinants are discussed, the properties being proved in the third order case, and the generalizations to the nth order indicated.

The illustrative examples throughout the book are well chosen and the exercises numerous and well graded. Finally, the printing is excellent, each page presenting an uncrowded appearance which should be inviting to the student and instructor alike.

R. A. BEAUMONT

NEW BOOKS RECEIVED

Le Superficie Algebriche. By Federigo Enriques. Nichola Zanichelli Editore, Bologne, 1949. 14+464 pp. 3000 lire.

Radio and Television Mathematics. By B. Fischer. New York, Macmillan, 1949. 18+484 pp. $6.00.

Transformation Calculus and Electrical Transients. By S. Goldman. New York, Prentice-Hall, 1949. 14+439 pp. $6.25.

Advanced Calculus for Engineers. By F. B. Hildebrand. New York, Prentice-Hall, 1949. 14+594 pp. $6.00.

Teorica Della Capitalizzazione. By F. Insolera. Turin, Giulio Einaudi, 1949. 237 pp. L. 1500.

Statistical Methods in Research. By P. O. Johnson. New York, Prentice-Hall, 1949. 18+377 pp. $5.00.

Rinehart Mathematical Tables. Compiled by H. D. Larsen. Alternate Edition. New York, Rinehart, 1948. 8+160 pp. $1.00.

Analytic Geometry and Calculus. By F. H. Miller. New York, Wiley, 1949. 12+658 pp. $5.00.

Differential Equations. Second Edition. By H. W. Reddick. New York, Wiley, 1949. 10+288 pp. $3.00.

First-year Mathematics for Colleges. By P. R. Rider. New York, Macmillan, 1949. 16+714 pp. $5.00.

Introduction to Algebraic Geometry. By J. G. Semple and L. Roth. Oxford University Press, 1949. 16+446 pp. $7.50.

Modern Algebra. Vol. 1. Translated from the second revised German edition by Fred Blum with revisions and additions by the author, B. L. van der Waerden. New York, Frederick Ungar, 1949. 12+264 pp. $5.50.

Quantum Theory of Fields. By G. Wentzel. Translated from the German by C. Houtermans and J. M. Jauch. New York, Interscience, 1949. 10+224 pp. $6.00.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

CLUB REPORTS, 1948–49

Pi Mu Epsilon, University of Missouri

The program of the *Missouri Alpha* Chapter of *Pi Mu Epsilon* for the academic year 1948–49 was:

Functions defined by more than one equation, by Dr. G. M. Ewing

Mathematics as a career, by Dr. L. M. Blumenthal

The students' points of view on mathematics courses, by a panel of students

Saturn and the Moon, by Dr. E. S. Hayner of the Astronomy department

Mathematics and vagueness, by Dr. W. D. Oliver, of the Philosophy department.

Other activities of the fraternity were: the annual initiation; the "Problem of the Month" Contest, consisting of a monthly mathematics problem with all prizes awarded to the first person giving the correct solution; and the annual calculus examination with prizes won by William Posladek, William Nichols, and Aldo Linsenbardt.

Newly elected officers are: President, William Nichols; Vice-President, Martin Smith; Secretary, Walter Hickman; Treasurer, Jesse Corder.

Kappa Mu Epsilon, College of St. Francis

The *Illinois Delta* Chapter of *Kappa Mu Epsilon* made a special study this year of the lives and works of the late Cassius J. Keyser and Alfred North Whitehead. Most of the papers and reports centered about these two outstanding scholars. The program was as follows:

Pastures of Wonder, by Keyser, by Sister M. Claudia O.S.F.

The life of Alfred North Whitehead, by Miss Mary Hodor

Cassius J. Keyser, by Miss Florence Kane

Humanization of the teaching of mathematics, by Miss Mary Jean Lafond

Educational ideals most worthy of loyalty, by Miss Lillian Rafac

Nature and Life, by Whitehead, by Sister M. Rita Clare O.S.G.

Whitehead's Introduction to Business Adrift, by Donham, by Miss Anne Hutchings

Aims of education, by Sister M. Perter O.S.F.

Mathematical curriculum, by Sister M. Hilary O.S.F.

The place of the classics in education, by Sister M. Elizbeth O.S.F.

Element in the history of thought, by Miss Jane Rourke

Non-euclidean geometries, by Miss Rita Grogan.

Miss Jane Rourke was elected by the Chapter to speak at a Mathematics Symposium sponsored by the Catholic Colleges of the Chicago area and held at DePaul University. Miss Rourke reviewed the book *Cybernetics* by N. Weiner. The paper was very timely and worth while.

The new officers for 1949–50 are: President, Mary Hodor; Vice-President, Anne Hutchings; Secretary, Margaret Dreska; Treasurer, Rita Grogan; Corresponding Secretary, Sister M. Rita Clare O.S.F.

Mathematics Club, University of Colorado

Members of the *Mathematics Club* of the University of Colorado presented the following talks during 1948–49:

On the origin of some mathematical terms and symbols, by Gideon Culpepper

How to win a free beer (nim), by Gordon Burton

Bromwich's integral, by Roy Reeves

A four number game, by Burrowes Hunt. Mr. Hunt extended the problem to include all real numbers

On p-adic numbers, by Prof. Burton Jones

Generating functions, by Prof. David Hawkins
On Jacobian elliptic functions, by Gideon Culpepper
On the algebra of sets, by David De Vol
On exterior differential forms, by Prof. Wesley Brittin
Cryptography, by Burrowes Hunt
Ruler and/or compasses, by Ralph Merrill
On some mathematical concepts in biophysics, by William Musgrove
Iteration, by Prof. Aubrey Kempner
Continued fraction approximation, by Prof. Burton Jones.

Officers for 1948–49 were: President, David De Vol; Secretary, Gordon Burton.

Sigma Phi Mu, Montclair Teachers College

Papers presented to *Sigma Phi Mu*, of Montclair Teachers College, New Jersey during the year 1948–49 were:

Cryptography, by Dr. Davis
Mathematics in the modern world, by Dr. Frieschower
Unknown quantity or logical system in algebra, by Dr. Fehr.

Movie strips shown were:

Geometry and Nature
Geometry in the Home
Seeing Is Believing.

The social program consisted of our annual Christmas Party with *Kappa Mu Epsilon*, a most enjoyable Probability Night, and our annual May picnic.

Instead of a February meeting here, several members attended the meeting of the *David Eugene Smith Mathematics Club* of Teachers College, Columbia University, at which time Mr. Clifford gave a talk on *Quality control*. Members of the Teachers College returned the visit to help us enjoy Probability Night.

We had a good attendance at all of our meetings. The number present each month averaged about sixty-five.

Officers for the coming year: President, William Koellner; Vice-President, Martin Vallaster; Secretary, Catherine Bruce; Treasurer, Dorothy Ryan.

Mathematics Club, University of Massachusetts

The *Mathematics Club* of the University of Massachusetts was organized on a formal basis at the beginning of the year and held monthly meetings. The following talks were presented to the Club:

Applications of probability, by Jerome Landry
Complex numbers, by Louis Robinson
Aerial navigation simplified, by Deane Beytes
Mathematical induction, by Murray Altsher
The mathematical analysis of a game, by Prof. I. H. Rose
The Ceasaro curve, by Francis Vigneau
Nature of postulational systems, by Janice Rittenburg

Infinite products, by Paul Ritger
The Lorentz transformation, by Robert San Soucie
Pythagorean numbers, by Henry Skillings
Number systems, by Walter Szetela
Mathematical principles of field artillery methods, by Stephen Allen
Symbolic logic, by Arthur Budd
Some entertaining applications of topology, by Prof. A. E. Andersen.

The newly elected officers are: President, Francis Vigneau; Vice-President, Charles Reynolds; Secretary-Treasurer, Grace Hyder; Faculty Advisor, Prof. A. E. Andersen.

Delta Rho, Southern Illinois University

Delta Rho, honorary mathematics fraternity of Southern Illinois University, held two meetings each quarter for the year 1948–49. Among the interesting papers given during the year were:

The four-color problem, by William Staudacher
The life of Pascal, by Joseph Rezetka
The life of LaGrange, by Louis Rowell
Philosophy and mathematics, by Dr. Charles Tenny
Abstract equivalence relations, by Charles Ablett
Involution and evolution, by Miss Ruth Sprankel
Zero, by Emmett Moll
Poly-dimensional manifolds, by Robert Etherton.

The outstanding meeting of the year was the annual Founders' Day banquet at which time nine new members were initiated. The total number initiated this year was 31.

The newly elected officers for the year 1949–50 are: President, William Staudacher; Vice-President, Charles Elliott; Secretary, Richard Vorwald; Treasurer, Imogene Beckemeyer; Program Chairman, Joseph Rezetka.

Pi Mu Epsilon, University of Oklahoma

The *Oklahoma Alpha* chapter of *Pi Mu Epsilon* began its activities for the year with a business meeting, at which time the following officers were elected for the year: Director, Lowell Gregory; Vice-Director, Richard Yeilding; Secretary-Treasurer, Gerald Webster.

The dates, speakers, and their topics for the subsequent meetings are as follows:

The five-color conjecture, by Dr. Arthur Bernhart
The effects of atmospheric disturbances on projectiles, by Otis Spears
Eigenfunctions and limits to the characteristic roots of a matrix, by Lowell Gregory
Cybernetics, by Dr. Carlton Berenda
Galois theory, by Miss Ann Sheldon.

The annual banquet and initiation was held at which Dr. J. Rud Nielsen, Research Professor of Physics, spoke on *The Raman effect*. Prizes consisting of mathematics books were presented to the winners of the Annual Mathematics Contest. These were: First prize, Mr. Miles Maxwell, Jr.; Second prize, Mr. Earl Scott.

The Officers for the year 1949–50 will not be elected until next fall.

Mathematics Club, Butler University

The *Mathematics Club* of Butler University submits the following report for the academic year 1948–49:

The activities included bi-weekly business meetings and programs and three social functions—Fall Wiener Roast, Christmas Party, and Spring Picnic.

The papers presented include:

Mathematical fallacies, by Mrs. Juna L. Beal

Mathematics in art, by Nancy Heemstra

Calculating machines, by Robert Gambill

Book review of Whom the Gods Love, by Mary Ethel Jackson

The abacus, by Clifford Wagoner

The navigation of the battleship Missouri from Norfolk, Virginia to Guantanamo, Cuba, by Dr. Harry E. Crull

Geometric exercises in paper folding, by Donald Reisinger

Mathematics of the automobile, by Max Sayler

The history of mathematics, by Anna White

The philosophy of mathematics, by Elsie Popplewell

Rubber sheet geometry, by Miss Jane Uhrhan

Mathematical literature, by Byron Carmichal.

The officers for the first semester 1949–50 are: President, Robert Gambill; Vice-President, Mary Ethel Jackson; Secretary, Phyllis Stultz; Treasurer, Ann Montgomery.

Kappa Mu Epsilon, University of New Mexico

The following papers were presented to the *New Mexico Alpha* chapter of *Kappa Mu Epsilon* for the year 1948–49:

History of the theory of planetary motion, by Robert Flanagan

The theory and use of the electronic oscilloscope, by Thomas Martin

The morality of science, by Dr. Sherman Smith

Bessel's functions applied to the theory of heat transfer by the use of fins, by Edward Rightley.

The general theory of alignment and contour charts, by Dr. Morris Hendrickson.

Science and art, by Dr. W. P. Albrecht.

Officers elected for 1949–50 are: President, Ross Schmidt; Vice-President, Thomas Wooten; Secretary, George Vincent; Treasurer, Prof. H. P. Rogers; Faculty Sponsor, Prof. Morris Hendrickson.

Mu Alpha Theta, Yeshiva College

The Yeshiva Mathematics Club, *Mu Alpha Theta*, was formed early in 1948. Since its inception, more than thirty papers have been presented to its members by students. In addition to major papers, there were some minor talks, including discussions on a number of problems which have appeared in the MONTHLY.

Other activities of the club included two parties, the presentation of a play at the College Class-Nite, and the planning of a Publication which later developed into a Yeshiva College Academic Journal.

The present officers are: President, William Frank; Vice-President, Seymour Haber; Secretary, Azriel Rosenfeld.

Kappa Mu Epsilon, Hofstra College

Regular monthly meetings were held by the *New York Alpha* Chapter of *Kappa Mu Epsilon* during 1948–49. Speakers and the titles of their papers were as follows:

Nomograms, by Leonard Hinder

Introduction to vector analysis, by Dr. L. F. Ollmann

Projectiles and parabolae, by Frank Hawthorne

Elements of Lewis Carroll, by Dorothea Reiffel

Huygen's relations with France, by Dr. Henri Brugmans of the Department of French

The useful and the useless in mathematics, by Dr. E. R. Stabler

Models and methods, by Lt. Col. Robert Yates of West Point, in which models for the construction of plane curves, diagrams, and linkages were presented.

The chapter members joined the Adelphi College Mathematics Club in hearing two lectures: *Diagrams and models*, by Col. Robert Beard; and *The number e and the law of continuing growth*, by Dr. Howard Fehr of Columbia University.

Two student and two faculty members attended the National Convention at Topeka, Kansas in April. Miss E. Marie Hove was re-elected National Secretary and Dr. L. F. Ollmann was re-elected National Treasurer of Kappa Mu Epsilon.

Pi Mu Epsilon, University of Kansas

The *Kansas Alpha* Chapter of *Pi Mu Epsilon* reports the following activities:

A get-acquainted picnic early in the fall term for all members of the chapter and all members of the mathematics staff.

A tea in January at which 13 new members were initiated.

Annual banquet in May at which Dr. N. W. Storer, of the Physics department, spoke on *Consider the universe.*

Officers for 1949–50 are: Director, Otho M. Rassmussen; Vice-Director, Martha Wagner; Corresponding Secretary, Wealthy Babcock; Recording Secretary, Frances Wolfe; Treasurer, Kathleen O'Donnell; Librarian, Gilbert Ulmer.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

ACKNOWLEDGMENT

The editor of the MONTHLY wishes to make grateful acknowledgment of the services rendered by the following persons who have refereed papers: A. A. Albert, C. B. Allendoerfer, H. T. R. Aude, J. M. Barbour, E. T. Bell, Lipman Bers, L. M. Blumenthal, P. W. Bridgman, J. R. Britton, Herbert Busemann, W. B. Carver, Alonzo Church, C. J. Coe, I. S. Cohen, J. L. Coolidge, A. H. Copeland, H. S. M. Coxeter, P. H. Daus, J. B. Diaz, Arnold Dresden, F. G. Dressel, P. S. Dwyer, J. D. Elder, Wade Ellis, H. W. Eves, C. D. Firestone, J. S. Frame, Abraham Franck, E. T. Frankel, Philip Franklin, G. E. Hay, E. Marie Hove, Ralph Hull, P. M. Hummel, R. D. James, R. A. Johnson, B. W. Jones, P. S. Jones, Edward Kasner, V. L. Klee, Jr., S. C. Kleene, R. E. Langer, H. D. Larsen, C. C. MacDuffee, Morris Marden, N. H. McCoy, Karl Menger, D. C. Murdoch, C. D. Olds, Gordon Pall, W. V. Parker, G. P. Price, G. Y. Rainich, E. D. Rainville, W. T. Reid, G. deB. Robinson, Louise J. Rosenbaum, Arthur Rosenthal, Vera Sanford, Robert Schatten, I. J. Schoenberg, Wladmir Seidel, Samuel Selby, I. M. Sheffer, W. H. Simons, F. C. Smith, Ruth B. Smyth, Pauline Sperry, Robert Stalley, F. H. Steen, Gabor Szegö, H. S. Uhler, Henry Van Engen, R. W. Wagner, R. J. Walker, J. L. Walsh, A. R. Williams, John Williamson, H. E. Wolfe, R. C. Yates, Leo Zippin.

BENJAMIN PEIRCE INSTRUCTORSHIPS AT HARVARD UNIVERSITY

The Mathematics Department of Harvard University expects to have one or two vacancies in the rank of Benjamin Peirce instructor beginning in the fall of 1950. Appointments are for a maximum period of three years. Applications should reach the Department prior to March 15, 1950.

EXECUTIVE DIRECTOR OF THE AMERICAN MATHEMATICAL SOCIETY

The Council and Trustees of the American Mathematical Society take pleasure in announcing the appointment of Dr. H. M. MacNeille as Executive Director of the Society. Dr. MacNeille assumed his duties in the New York office of the Society on November 15, 1949. The position of Executive Director to aid in the administration of the affairs of the Society was created by the Council and approved by the Trustees, on recommendation of a Committee on Reorganization. The members of the Society approved this action of the Council when the new By-Laws were adopted at the Annual Meeting in December, 1948.

Dr. MacNeille was graduated with highest honors from Swarthmore College in 1928. For two years he was Rhodes Scholar from New Jersey at Balliol College, Oxford, and took an honors degree in mathematics at that university. He received the degree of Doctor of Philosophy from Harvard University in 1935, the subject of his dissertation being *Extensions of partially ordered sets*. During the academic year 1935–1936 he was Sterling Fellow at Yale University, from 1936–1938 he was an Instructor of Mathematics at Harvard University, and from 1938 to 1947 he was Professor of Mathematics at Kenyon College. At the close of the Army and Air Force training programs at Kenyon College early in 1944 Dr. MacNeille was appointed Scientific Liaison officer and later head of the London Mission of the Office of Scientific Research and Development. In 1946 he became the Scientific Director of the newly established Office of the Assistant Naval Attache for Research in London. Since January, 1948 he has been Chief of the Fundamental Research Branch of the United States Atomic Energy Commission in Washington.

PERSONAL ITEMS

Professor A. T. Brauer of the University of North Carolina has received one of the three inaugural Science Research Awards of the Oak Ridge Institute of Nuclear Studies for "significant contributions to science in the South."

Dean E. L. Mackie of the University of North Carolina represented the Association at the installation of President A. H. Edens of Duke University on October 22, 1949.

Dr. C. V. Newsom, Assistant Commissioner for Higher Education for the State of New York, was the representative of the Association upon the commemoration of the 125th Anniversary of Rensselaer Polytechnic Institute on October 12–15, 1949.

Professor J. B. Reynolds of Lehigh University served as the representative of the Association at the inauguration of President D. E. Weinland of Moravian Seminary and College for Women on October 22, 1949.

Dr. S. A. Schelkunoff of the Bell Telephone Laboratories was the delegate of the Association at a meeting of the American Standards Association on October 14, 1949.

Assistant Professor Edith R. Schneckenburger of the University of Buffalo represented the Association at the installation of the Very Reverend Juvenal Lalor as president of St. Bonaventure College on September 22, 1949.

Professor J. M. Thomas, Duke University, was appointed to represent the Association at the inauguration of President A. G. Moron of Hampton Institute on October 29, 1949.

Montana State University announces: Assistant Professor Harold Chatland of Ohio State University has been appointed to a professorship; Mr. R. L. Berggren, University of Minnesota, has been appointed to an instructorship.

The Institute for Mathematics and Mechanics of New York University reports: Professor Emil Artin of Princeton University, Professor Hans Rade-

macher of the University of Pennsylvania, and Professor Carl Siegel of The Institute for Advanced Study have been appointed Visiting Professors; Dr. Bernard Friedman, who is a research associate at the Institute, has been promoted to an associate professorship at Washington Square College.

At The Oklahoma Agricultural and Mechanical College the Department of Mathematics is now offering the Ph.D. degree. This expansion is being made with the support of a research contract with the Office of Naval Research. The work of this contract is under the direction of Research Professor Nachman Aronszajn. The following are also participating in this work: Professors A. H. Diamond, O. H. Hamilton, H. W. Smith; Research Assistants M. Arnsten, C. J. Clark, Alan Jennings, L. F. McAuley, G. B. Pedrick, G. D. Pirtle, A. W. Wortham, Arthur Zeichner. The following research divisions are supporting the program: The Research Foundation, The Agricultural Experiment Station, The Division of Engineering Research and Experiment Station and The Oklahoma Power and Propulsion Laboratory.

The University of Buffalo makes the following announcements: Mr. Benjamin Lapidus of the University of Toronto has been appointed to an instructorship; Mr. A. G. Fadell and Mr. L. J. Montzingo have received appointments as teaching fellows.

University of California at Los Angeles reports: Professor J. B. Rosser of Cornell University, who has the position of Director of Research at the Institute for Numerical Analysis, has been appointed Visiting Professor of Mathematics; Assistant Professor F. A. Valentine has been promoted to an associate professorship and has been granted a year's leave of absence for the purpose of further study at the University of Chicago; Dr. L. J. Paige and Dr. J. D. Swift have been promoted to assistant professorships; Dr. P. G. Hodge, Jr. of Brown University has been appointed to an assistant professorship; Dr. G. M. Wing, Cornell University, has been appointed to an instructorship; Assistant Professor E. R. Worthington has retired.

University of South Carolina announces: Adjunct Professor Marguerite Z. Hedberg has been promoted to an associate professorship; Mr. C. W. Huff of Pennsylvania State College has been appointed to an instructorship.

Washington University announces the appointments of Dr. I. I. Hirschman, Harvard University, to an assistant professorship and of Dr. H. Margaret Elliott, Radcliffe College, to an instructorship.

Professor E. P. Adams of Princeton University has retired with the title of Emeritus Professor of Physics.

Professor V. W. Adkisson, chairman of the Department of Mathematics of the University of Arkansas, has been appointed Dean of the Graduate School.

Dr. A. V. Baez, formerly instructor in the Department of Physics at Stanford University, has accepted an appointment as a research physicist at the Cornell Aeronautical Laboratory, Buffalo, New York.

Mr. R. T. Bailey has been appointed to an instructorship at Norwich University.

Brother Ladislaus Balbert, formerly at Christian Brothers College, Memphis, is now an instructor at Colegio San Jose, Bluefield, Nicaragua.

Dr. R. E. Basye has been promoted to an associate professorship at Agricultural and Mechanical College, Texas.

Mr. Leon Benson of Lehigh University is teaching now at Blasdell High School, Buffalo, New York.

Professor H. L. Black of Michigan State College has accepted an appointment as associate professor in the Department of Engineering, Pennsylvania State College.

Associate Professor R. C. Blackwell has been promoted to a professorship at Furman University.

Dr. A. L. Blakers of Princeton University has been appointed to an assistant professorship at Lehigh University.

Dr. J. H. Blau, Massachusetts Institute of Technology, has received an appointment as assistant professor at Pennsylvania State College.

Associate Professor F. E. Bowling has been promoted to the position of Head of Department of Mathematics, Lincoln Memorial University.

Dr. L. E. Boyer has been promoted to the chairmanship of the Department of Mathematics, State Teachers College, Millersville, Pennsylvania.

Dr. R. A. Bradley has been appointed to an assistant professorship at McGill University.

Dean J. W. Branson, who has been serving as acting president of The New Mexico College of Agriculture and Mechanic Arts, has been appointed President.

Miss Jeneva J. Brewer, formerly graduate fellow at the University of Wichita, has been appointed to an instructorship.

Assistant Professor J. B. Brewster of Clemson Agricultural College has been promoted to an associate professorship.

Mr. B. B. Clark, previously graduate student at the University of Michigan, has been appointed to an instructorship at Grinnell College.

Instructor R. L. Coker, Southern College of Optometry, has been appointed Professor of Mathematics and Physics.

Assistant Professor R. H. Cole has been promoted to an associate professorship at the University of Western Ontario.

Assistant Professor Mamie M. Davis, Southeastern Louisiana College, has accepted an instructorship at Mississippi Southern College.

Mr. B. B. Dressler has been promoted to an assistant professorship at Kent State University.

Dr. J. C. Eaves has been appointed to an assistant professorship at the University of Alabama.

Mr. J. L. Ericksen has been appointed to a teaching fellowship at Indiana University.

Assistant Professor Walter Fleming, Fort Hays Kansas State College, is now

Associate Professor of Mathematics at Mankato State Teachers College, Minnesota.

Mr. Charles Franciol is teaching at Gillis High School, Lake Charles, Louisiana.

Dr. M. J. Gottlieb, who has been at the Institute for Advanced Study, has been appointed to an assistant professorship at the University of Chicago.

Miss Alice Graeber, who has been teaching at Dates School, San Francisco, has accepted a position as teacher at Mission High School, San Francisco.

Mr. H. H. Harman, formerly research associate, has been promoted to the position of Chief of the Statistical Research and Analysis Unit, Department of Army.

Mr. J. R. Hodges, teaching fellow at Tulane University, has received an appointment as instructor at Little Rock Junior College.

Mr. Julius Honig has the position of Research Assistant at Long Island College of Medicine.

Mr. G. K. Howe of Bell Aircraft Corporation, Marietta, Georgia, has retired.

Mr. Eugene Isaacson has been appointed to an assistant professorship at New York University.

Dr. John Kronsbein has been appointed Head of the Department of Engineering of Evansville College.

Dr. Paolo Lanzano of the University of Rome has received an appointment as instructor at St. Louis University.

Dr. Lee Lorch, City College of the City of New York, has been appointed to an assistant professorship at Pennsylvania State College.

Professor E. J. Lowry has been promoted to the position of Head of the Department of Mathematics, Hastings College.

Miss Hazel L. Mason, formerly instructor at Mary Hardin-Baylor College, is now a member of the faculty of the Grand Prairie Public Schools, Texas.

Professor J. C. C. McKinsey has returned to the Oklahoma Agricultural and Mechanical College after spending the last two years on leave with the Rand Corporation.

Dr. G. D. Mostow, who has been at the Institute for Advanced Study, has been appointed to an assistant professorship at Syracuse University.

Mr. E. J. Musch has been appointed to an instructorship at the University of Louisville.

Mr. J. A. S. Neilson has been promoted to the chairmanship of the Department of Mathematics, Westmont College.

Associate Professor H. E. Nelson of Gustavus Adolphus College has been appointed to an associate professorship at Augustana College.

Mr. J. A. Nickel is now serving as graduate assistant at Oregon State College.

Dr. Helen K. Nickerson has been promoted to an assistant professorship at Wheaton College, Massachusetts.

Mr. J. V. Pennington, formerly associate director of Southwest Research Institute, San Antonio, has been appointed Technical Director of Drilling Research Inc., Houston.

Associate Professor P. M. Pepper of the University of Notre Dame has been appointed Associate Professor of Industrial Engineering at Ohio State University.

Mr. C. R. Perisho has been promoted to an associate professorship at Nebraska Wesleyan University.

Professor Mary Pettus, Lander College, has been appointed to an assistant professorship at the University of Richmond.

Professor J. C. Polley of Wabash College has been promoted to the position of Head of the Department of Mathematics.

Associate Professor D. W. Pugsley has been promoted to the position of Professor and Head of the Department of Mathematics at Berea College.

Professor H. R. Pyle of Whittier College is now Head of the Department of Mathematics.

Assistant Professor G. E. Reves of The Citadel has been promoted to an associate professorship.

Professor B. D. Roberts, New Mexico Highlands University, has been promoted to the position of Dean and Head of the Department of Mathematics.

Miss Margarita E. Rodríguez, formerly at Montanzas Instituto, Cuba, is teaching at Instituto del Vedado, Cuba.

Assistant Professor D. R. Ryan of Gonzaga University is now Chairman of the Department of Mathematics.

Mr. A. H. Sarno, St. John's College, has been promoted to an assistant professorship and chairmanship of the Department of Mathematics.

Mr. F. W. Saunders has been promoted to a professorship at Coker College.

Mr. J. A. Schumaker of MacMurray College for Women has been appointed to an assistant professorship.

Mr. Whitney Scobert, formerly at University of Oregon, has accepted a position as associate professor at Idaho State College.

Mr. R. C. Seber, formerly graduate assistant at the State University of Iowa, has been appointed to an instructorship at Rockford College.

Mr. D. P. Shore is teaching at Carter-Riverside High School, Fort Worth.

Mr. C. E. Shotwell, who has been an assistant instructor at the University of Missouri, has received an appointment as instructor at the University of the South.

Mr. A. T. Skinner has been appointed to an assistant professorship at Champlain College.

Professor A. A. Smith, previously chairman of the Department of Mathematics of San Angelo College, has been appointed Business Manager of Texas Western College.

Associate Professor Ernst Snapper of the University of Southern California is now Visiting Associate Professor at Princeton University.

Assistant Professor P. I. Speicher has been promoted to an associate professorship at Albright College.

Mr. Isaay Stempnitzky is serving as research assistant in Electrical Engineering at Massachusetts Institute of Technology and teaching fellow at Brandeis University.

Mr. R. F. Steward, formerly teaching assistant at New Jersey College for Women, has been appointed to an instructorship at Virginia Military Institute.

Assistant Professor C. F. Strobel of North Carolina State College has been promoted to an associate professorship.

Mr. J. S. Stubbe of the University of New Hampshire has been appointed to an assistant professorship at Clark University.

Mr. Kaidy Tan is now Tutor at Anglo-Chinese College, Kulangsu, Amoy, China.

Professor Alex Tartler of Drexel Institute has been appointed to an associate professorship at Lafayette College.

Mr. B. P. Taylor is now Head of the Calculation Group, Lofting Department, Douglas Aircraft Company.

Mr. C. M. Terry has been appointed to a fellowship at Iowa State College.

Instructor O. M. Thomas of the United States Naval Academy has been promoted to an assistant professorship.

Mr. C. W. Trigg has been appointed Assistant to the Dean of Instruction, Los Angeles City College.

Dr. W. R. Utz, University of Michigan, has been appointed to an assistant professorship at the University of Missouri.

Miss Frances Weisbecker of Milwaukee-Downer College has been promoted to an assistant professorship.

Assistant Professor D. W. Whitford has been promoted to an associate professorship at Polytechnic Institute of Brooklyn.

Professor P. D. Wilkins of Bates College has been promoted to the position of Head of the Department of Mathematics.

Mr. A. D. Wirshup, formerly an instructor at Multnomah College, has accepted a teaching fellowship at Oregon State College.

Miss Ferna E. Wrestler has been promoted to an assistant professorship at the University of Wichita.

Miss Frances M. Wright has been promoted to an assistant professorship at Triple Cities College of Syracuse University.

Associate Professor A. J. Zanolar has been appointed President of St. Joseph's College, Indiana.

Mr. R. K. Zeigler has been appointed to an associate professorship at Bradley University.

Professor Emeritus S. E. Brasefield of Rutgers University died October 14, 1949.

Professor A. F. Carpenter, formerly executive officer of the Department of Mathematics of the University of Washington, died on October 16, 1949.

Professor Emeritus Peter Field of the University of Michigan, died on September 24, 1949. He was a charter member of the Association.

Professor F. A. Foraker of the University of Pittsburgh, died on June 30, 1949. He was a charter member of the Association.

Associate Professor R. K. Luneburg of the University of Southern California died on August 19, 1949.

Dr. D. A. Rothrock, professor emeritus of Indiana University and a charter member of the Association, died September 2, 1949.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 91 persons have been elected to membership by the Board of Governors on applications duly certified:

J. D. R. BAHNG, Student, St. Norbert College, West DePere, Wis.

ESTHER P. BARNHART, M.A.(Michigan) Instructor, Capital University, Columbus 9, Ohio.

R. D. BARTZ, Student, University of Wisconsin, Milwaukee 3, Wis.

R. W. BEALS, JR., M.Ed.(Alfred) Instructor, Alfred University, N. Y.

R. W. BIRD, B.A.(Roosevelt) Bookkeeper, Hartford Fire Insurance Co., Chicago, Ill.

W. W. BLAKELY, B.A.(Macalester) Grad. Asst., Macalester College, St. Paul, Minn.

I. H. BRUNE, Ph.D.(Ohio State) Asso. Professor, Iowa State Teachers College, Cedar Falls, Iowa.

R. K. BUTZ, B.S.(Colorado A & M) Grad. Student, University of Georgia, Athens, Ga.

W. W. CHAMBERS, M.A.(Colorado State) Instructor, Purdue University, Indianapolis, Ind.

M. H. CLAYTON, M.E.(North Carolina) Instructor, North Carolina State College, Raleigh, N. C.

L. W. COHEN, Ph.D.(Michigan) Professor, Queens College, Flushing, L. I.

J. W. COY, M.A.(New Mexico) Instructor, Michigan State College, East Lansing, Mich.

W. V. CRESSY, Student, University of California, Los Angeles, Calif.

W. F. CRUM, B.A.(Carleton) Instructor, Carleton College, Northfield, Minn.

DOROTHY S. CURRY, M.S.(Chicago) Asso. Professor, Wilberforce University, Ohio.

T. S. DEAN, M.S.(North Texas) President, Design Associates, Inc., Sherman, Texas.

V. E. DIETRICH, M.Sc.(Notre Dame) Grad. Asst., Purdue University, Lafayette, Ind.

D. W. DUBOIS, B.S.(Oklahoma) Grad. Asst., University of Oklahoma, Norman, Okla.

ARTHUR ERDELYI, D.Sc.(Edinburgh) Professor, California Institute of Technology, Pasadena, Calif.

LORRAINE W. FARBER, M.A.(Buffalo) Instructor, University of Buffalo, N. Y.

JEAN B. FEIDNER, B.A.(William Smith) Instructor, University of Buffalo, N. Y.

RUDOLF FEIGE, M.A.(Berlin) Instructor, University of Cincinnati, Ohio.

J. R. FLEMING, Captain, U.S.A.F., Air Weather Service, Warner Robins, Ga.

F. A. FOGG, B.S.(Chile) Engr., Collins Radio Co., Cedar Rapids, Iowa.

P. L. FORD, M.S.(Louisiana State) Instructor, Kemper Military School, Boonville, Mo.

J. H. FOUNTAIN, B.A.(Buffalo) Grad. Student, University of Buffalo, N. Y.

G. C. FRANCIS, M.S.(Minnesota) Instructor, Carleton College, Northfield, Minn.

W. M. FRANK, Student, Yeshiva University, Brooklyn, N. Y.

F. C. GERMAN, M.A. (Illinois) Asso. Professor, Kansas State Teachers College, Pittsburg, Kansas.

GLADYS E. GILLMAN, B.A. (Skidmore) Teacher, Winsor School, Boston, Mass.

R. D. GLAUZ, M.S. (Michigan) Instructor, Montana State College, Bozeman, Mont.

H. E. GOHEEN, Ph.D. (Stanford) Asst. Professor, Syracuse University, N. Y.

B. T. GOLDBECK, M.A. (Texas Christian) Student, Texas Christian University, Fort Worth, Texas.

BEULAH GRAHAM, M.A. (Kentucky) Teacher, Campbellsville College, Ky.

U. S. GREENE, C.P.A., Marine Midland Bldg., Binghamton, N. Y.

VIRGINIA M. HALL, A.M. (Boston Univ.) Instructor, Simmons College, Boston, Mass.

PARKER HAMILTON, B.S. (Harvard) Asst. Professor, Antioch College, Yellow Springs, Ohio.

C. L. HARBISON, M.A. (Indiana) Asst. Professor, South Dakota School of Mines and Technology, Rapid City, S. D.

GERALD HARRISON, Ph.D. (Calif. Inst.) Asst. Professor, Wayne University, Detroit, Mich.

M. J. HELLMAN, M.S. (CCNY) Instructor, Rutgers University, Newark, N.J.

B. O. HOYLE, M.S. (Southern Illinois) Instructor, Southern Illinois University, Carbondale, Ill.

W. R. HYDEMAN, A.M. (Syracuse) Mathematician, Navy Department, Washington, D. C.

EUGENE ISAACSON, Ph.D. (New York) Instructor, New York University, N. Y.

P. W. M. JOHN, M.A. (Oxon) Instructor, University of Oklahoma, Norman, Okla.

G. P. JOHNSON, M.A. (Minnesota) Grad. Student, University of Minnesota, St. Paul, Minn.

R. P. KANIA, M.A. (New Zealand) Lecturer, Teachers Training College, Auckland, N. Z.

C. E. KELLEY, B.S. (Central Missouri) Asst. Instructor, University of Missouri, Columbia, Mo.

D. E. KIBBEY, Ph.D (Illinois) Asso. Professor, Syracuse University, N. Y.

JOSEPH KLEIN, B.S. (Rutgers) Student, Rutgers University, New Brunswick, N. J.

R. J. KOHLMEYER, M.A. (Washington) Instructor, Pratt Institute, Brooklyn, N. Y.

G. M. KUZNETS, Ph.D. (California) Asso. Professor, University of California, Berkeley, Calif.

VIVIAN E. LARSON, M.A. (Wisconsin) Instructor, University of Wisconsin, Milwaukee, Wis.

F. W. LOTT, JR., M.A. (Michigan) Asst. Professor, Iowa State Teachers College, Cedar Falls, Iowa.

R. E. LOWNEY, M.A. (Michigan State) Asst. Professor, Montana State College, Bozeman, Mont.

J. A. MARTIN, SR. Partner, National Oil Co., Lisbon, Ohio.

H. C. MCKENZIE, M.A. (Wisconsin) Instructor, University of Colorado, Boulder, Colo.

GENEVIEVE T. MEYER, M.A. (Marquette) Instructor, University of Wisconsin, Milwaukee, Wis.

MICHAEL MORELLI, S.B. (M.I.T.) Air Material Command, Cambridge Field Station, Mass.

F. J. MURRAY, Ph.D. (Columbia) Professor, Columbia University, New York, N. Y.

R. E. NELSON, M.A. (Dartmouth) Asso. Professor, Dickinson College, Carlisle, Pa.

E. N. NILSON, Ph.D. (Harvard) Asst. Professor, Trinity College, Hartford, Conn.

CHESTER PACHUCKI, M.S. (De Paul) Instructor, De Paul University, Chicago, Ill.

L. J. PAIGE, Ph.D. (Wisconsin) Instructor, University of California, Los Angeles, Calif.

A. J. PEJSA, M.S. (Marquette) Asst. Professor, U. S. Naval Academy, Annapolis, Md.

C. L. PERRY, JR., Ph.D. (Michigan) Asst. Professor, University of Arkansas, Fayetteville, Ark.

M. B. PORTER, M.A. (Texas) Asst. Professor, Southwest Texas State Teachers College, San Marcos, Texas.

R. E. PRIEST, Student, University of Illinois, Urbana, Ill.

N. J. ROSE, M.E. (Stevens) Instructor, Stevens Institute of Technology, Hoboken, N. J.

R. M. ROSS, M.A. (Indiana) Asst. Professor, Rose Polytechnic Institute, Terre Haute, Ind.

J. P. ROTH, M.S.(Michigan) Student, University of Michigan, Ann Arbor, Mich.

BARNEY SANDLER, B.A.(Brooklyn) Instructor, Institute of Applied Arts and Science, Brooklyn, N. Y.

W. L. SAUTER, Student, University of Louisville, Ky.

R. W. SCHMIED, B.A.(Texas) Instructor, Tulane University, New Orleans, La.

. A. SCHUMAKER, M.A.(Illinois) Asst. Professor, MacMurray College for Women, Jacksonville, Ill.

SISTER MARY PETRONIA, Ph.D.(Notre Dame) Teacher, Mount Mary College, Milwaukee, Wis.

SISTER MIRIAM FRANCIS, Ph.D.(Catholic) Instructor, Xavier University, New Orleans, La.

R. D. STALLEY, M.A.(Oregon) Instructor, University of Arizona, Tucson, Ariz.

E. A. STAVINOHA, B.S.(Baylor) Grad. Fellow, Oklahoma A & M College, Stillwater, Okla.

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THE APRIL MEETING OF THE MISSOURI SECTION

The Missouri Section of the Mathematical Association of America met in Mumford Hall on the campus of the University of Missouri in Columbia, on Saturday, April 9, 1949. Professor P. R. Rider, Chairman of the Section, presided at the morning session, and Professor P. B. Burcham presided at the afternoon session.

Seventy-four persons were in attendance including the following twenty-seven members of the Association: L. M. Blumenthal, C. H. Brown, P. B. Burcham, Mary L. Cummings, W. C. Doyle, R. E. Ekstrom, G. M. Ewing, C. V. Fronabarger, W. L. Graves, Nola L. Haynes, F. F. Helton, R. E. Hogan, L. O. Jones, C. W. Mathews, J. M. Marr, Karl Menger, E. F. Moore, A. D. Pierson, P. R. Rider, J. S. Rosen, J. W. Sawyer, C. E. Shotwell, R. L. Snider, W. L. Stamey, A. D. Talkington, Margaret F. Willerding, and W. D. Williams.

The following officers were elected for the coming year: Chairman, C. W. Mathews, Washington University; Vice-Chairman, G. H. Jamison, Northeast Missouri State Teachers College; Secretary-Treasurer, Margaret F. Willerding, Harris Teachers College. At the conclusion of the meeting tea was served in the lounge of Gwynn Hall.

The Section had Professor Karl Menger of the Illinois Institute of Technology as a guest speaker. Professor Menger spoke on *Some Aspects of the Concept of Convexity*. The following papers were also presented:

1. *An approximate solution of Stieltjes integral by use of Cantelli's inequality*, by Dr. Maria Castellani, University of Kansas City, introduced by the Secretary.

2. *The metrization of torsion*, by Dr. L. M. Blumenthal, University of Missouri.

The paper presents a metric definition of torsion of a metric space at an accumulation point p based merely upon the consideration of the mutual distances of points of quadruples. In distinction to the definition given by Alexits (*Compositio Mathematicae*, vol. 6 (1938–39), pp. 471–477) the new definition is (1) symmetric in all distances, and (2) avoids iterated limits. It is shown that the metric torsion so defined yields the classical torsion for curves of the kind usually dealt with in classical differential geometry.

3. *A reorganization of general mathematics of colleges*, by Dr. Margaret F. Willerding, Harris Teachers College.

Mathematical education is in a sad state of confusion. General mathematics, which is often all the mathematics a student gets in high school, is poorly conceived and inadequately presented. There is great need in this field for better organization and for better teaching. The great organizations of mathematicians and teachers of mathematics should present a united front for improvement to state officers of education and to others who plan the educational offering in the high school.

4. *The trends in mathematical education in high schools*, by Prof. G. H. Jamison, Northeast Missouri State Teachers College.

Every year more and more students who have had only one year of general secondary mathematics are entering colleges. These students are totally unprepared to enter the college freshman mathematics courses in general mathematics as they are now organized. A college course in general mathematics must be organized for these students. The course must supplement the general mathematics courses given in the secondary schools. It must be primarily designed for students who signify that they have no intention of taking more than one year of mathematics. The objectives of such a basic first year course must center around the needs—both present and future—of the college student. It must give the student a better grasp of mathematics as a whole and of the interdependence of its various parts, and accustom him to use, in later applications, the method best adapted to the problems at hand. In order to meet these objectives the content of the course must be studied on an elaborate scale, and the subject matter picked with care. The problem of the reorganization of general mathematics is a fertile field for research and experimentation, and is a challenge to every mathematician and educator.

5. *Use of mnemonic devices in mathematics*, by C. W. Mathews, Washington University.

In teaching mathematics, mnemonic devices are used at times when it is easier to learn the facts from the fundamentals. An example served as an illustration of the truth of this statement. An effective mnemonic device must be one which is based upon well known facts, and should be simple. It should not admit exceptions, or hamper generalizations of the concepts or procedures. Students can learn better with a personalized memory aid than with a poor "mail order" variety. Consider a mnemonic device carefully before you use it.

CHAS. W. MATHEWS, *Secretary-Treasurer*

THE APRIL MEETING OF THE UPPER NEW YORK STATE SECTION

The annual meeting of the Upper New York State Section of the Mathematical Association of America was held at the University of Buffalo, Buffalo, New York, on Saturday, April 30, 1949. The Chairman of the Section, Professor E. B. Allen of Rensselaer Polytechnic Institute, presided at the morning session; the Vice-Chairman, Professor W. H. Durfee of Hobart College, presided at the afternoon session. At the conclusion of the afternoon session tea was served.

One hundred twenty-three persons attended the meeting, including the following seventy members of the Association: Janet E. Abbey, E. B. Allen, H. P. Atkins, H. T. R. Aude, H. W. Baeumler, G. B. Banks, V. N. Behrns, R. L. Beinert, W. W. Bessell, I. S. Boak, A. L. Buchman, C. G. Burger, F. J. H. Burkett, I. S. Carroll, H. S. M. Coxeter, F. F. Decker, E. J. Downie, Walter H. Durfee, F. P. Egan, J. E. Freund, H. M. Gehman, B. H. Gere, R. D. Gordon, Lillian Gough, N. G. Gunderson, May N. Harwood, M. B. Haslam, Margaret W. Hickman, R. G. Hill, H. E. Hoffman, A. W. Jones, C. E. Lemke, Caroline A. Lester, J. V. Limpert, R. K. Longley, R. R. R. Luckey, Dis Maly, E. W. Marchand, June M. McArtney, Rudolph Meyer, F. E. Milliman, Harriet F. Montague, Mabel D. Montgomery, D. S. Morse, Abigail M. Mosey, C. W. Munshower, C. R. Newell, Ruth B. Noller, B. C. Patterson, Theresa L. Podmele, J. F. Randolph, P. C. Rapp, C. E. Rhodes, P. J. Schillo, Edith R. Schneckenburger, L. F. Scholl, B. B. Sharpe, William A. Smith, Ruth W. Stokes, D. D. Strebe, Mary C. Suffa, R. T. Tear, G. W. Walker, J. F. Wardwell, F. C. Warner, E. T. Welmers, Ina W. Welmers, Mary E. Williams, Frances M. Wright, R. C. Yates.

The following officers were elected: Chairman, Walter H. Durfee, Hobart College; Vice-Chairman, B. C. Patterson, Hamilton College; Secretary, C. W. Munshower, Colgate University. The Chairman was authorized to appoint a committee to study the relation between secondary school and college mathematics in New York State. It was announced that E. B. Allen had been elected Sectional Governor. The 1950 meeting will be held at Syracuse University in the spring of 1950.

On Friday evening, April 29, 1949, Professor H. S. M. Coxeter of the University of Toronto delivered a lecture on the Fenton Foundation of the University of Buffalo on the subject *Arithmetical Games and Phyllotaxis*; many members of the Association attended.

The following papers were presented:

1. *The excenters of a hyperbolic triangle*, by Professor H. S. M. Coxeter, University of Toronto.

Since the angle bisectors of a triangle ABC are the loci of points equidistant from two sides, any intersection of two bisectors lies also on a third; therefore, the six bisectors are the sides of a complete quadrangle. This simple argument is valid in elliptic geometry as well as in euclidean. But in hyperbolic geometry it fails to cover the case when the external bisectors of two of the angles are parallel or ultraparallel. With the aid of coordinates it is readily proved that the internal bisector of the third angle still belongs to the same pencil (of parallels or ultraparallels), so that

there is an escribed horocycle or hypercycle. A synthetic proof is obtained by exhibiting these three bisectors as the diagonals of a Brianchon hexagon whose sides are the tangents to the absolute conic at the six "ends" of the lines BC, CA, AB, taken in a suitable order (different from the natural order used for the concurrence of internal bisectors in Coxeter's *Non-Euclidean Geometry*).

2. *The problem of eight points*, by Professor W. B. Carver, Cornell University.

Because of illness Professor Carver was unable to be present; his paper will be published in this MONTHLY.

3. *The addition formulas of trigonometry*, by Professor J. F. Randolph, University of Rochester.

A student will grant that two points on the rim of a wheel remain the same distance apart regardless of how the wheel is turned, and must therefore grant, for A and B arbitrary angles, that

$$\cos (A - B) = \cos A \cos B + \sin A \sin B. \tag{1}$$

It is only necessary to consider the points represented by $(1, 0)$, $(\cos [A-B], \sin [A-B])$, $(\cos B, \sin B)$ and $(\cos A, \sin A)$, note that the distance between the first pair (seen by rotating the first pair along the unit circle through the angle B), express the equality of these distances by using the distance formula, and simplify. Since (1) holds without restrictions on the angles A and B, and the values of the trigonometric functions for quadrantal angles are known, we set $A=0$ in (1) to obtain $\cos (-B)=\cos B$; set $A=90°$ in (1) to obtain $\cos (90°-B)=\sin B$ and hence also $\cos B=\sin (90°-B)$; and from periodicity note $\sin (-A)=\sin (360°-A)=\sin (90°-(A-270°))$ $=\cos (A-270°)=-\sin A$ (the last equality following from (1) with $B=270°$). Thus the sum and difference formulas and all consequences of them (including reduction formulas) are easily obtained without apology.

4. *An algebraic algorithm*, by Dr. N. G. Gunderson, University of Rochester.

The speaker described Rosser's algorithm for solving the linear Diophantine equation, and discussed several applications.

5. *Mathematics in New York State Institutes*, by Professor I. S. Boak, New York State Agriculture and Technical Institute, Canton, New York.

The purpose and scope of mathematics courses was discussed in relation to the technological courses offered at the Institutes. The influence of special requirements of technical courses on methods of teaching and content, and the provisions for variations in ability and mathematical preparation during the early weeks of the course was indicated. The work of the New York State curriculum workshop in relation to mathematics was described, and trends in this field were discussed. The paper concluded with a brief discussion of trends toward standardization of courses and testing programs.

6. *Flying inertias*, by Dr. E. T. Welmers, Bell Aircraft Corporation.

The determination of the inertia properties of aircraft or aircraft components furnishes an interesting application of the formulas developed in elementary calculus. The moments of inertia of the aircraft as a whole are essential in determining the flight path. In the design of control surfaces to prevent flutter, it is necessary to determine the static balance (dependent on the simple moment), moment of inertia about a hinge line, and the dynamic balance (dependent on products of inertia). Illustrations of these requirements were given for unconventional control surfaces involving, particularly, sweep and dihedral. More advanced mathematics is required to evaluate the virtual inertia of the air which surrounds an oscillating wing or the moment of inertia of liquid fuel. Finally, mention was made of the mathematical bases for certain experimental methods of determining inertias.

7. *Extra-sensory perception, a problem in the theory of confirmation*, by Professor J. E. Freund, Alfred University.

Professor Freund discussed the application of the frequency theory of probability to the concept of the confirmation of scientific hypotheses. The probability of a scientific theory was defined in terms of its associated class of observable phenomena, and a formula was derived to show how the number and variety of instances of confirmation effect the degree to which a theory may be considered to be confirmed. These results were applied to the problem of confirming an "unusual" theory, e.g. the theory of extra-sensory perception, and it was demonstrated why theories are not accepted at times in spite of the fact that the positive evidence seems overwhelming. The danger of the statement "It couldn't have happened by chance" was discussed with reference to this problem.

C. W. MUNSHOWER *Secretary*

THE APRIL MEETING OF THE OHIO SECTION

The thirty-third annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio on Saturday, April 2, 1949. Professor R. H. Marquis, Chairman of the Section, presided.

Ninety-two persons registered attendance, including the following sixty-five members of the Association: W. E. Anderson, Max Astrachan, Grace M. Bareis, I. A. Barnett, H. M. Beatty, Henry Blumberg, J. B. Brandeberry, Foster Brooks, C. D. Calhoon, V. B. Caris, E. H. Clarke, W. F. Cornell, Rufus Crane, Grace Cutler, Wayne Dancer, Violet Davis, R. H. Downing, B. B. Dressler, P. L. Evans, H. E. Fettis, B. C. Glover, L. J. Green, Marshall Hall, Jr., Frances Harshbarger, C. H. Heinke, P. S. Herwitz, Carl Holtom, S. J. Jasper, E. D. Jenkins, Margaret E. Jones, Chosaburo Kato, L. C. Knight, L. L. Lowenstein, W. C. Lowry, R. H. Marquis, H. R. Mathias, E. J. Mickle, L. H. Miller, Max Morris, H. C. Parrish, H. S. Pollard, Irene Price, Tibor Rado, S. E. Rasor, O. W. Rechard, F. W. Reed, P. V. Reichelderfer, S. A. Rowland, K. C. Schraut, C. E. Sealander, J. L. Solomon, V. C. Stechschulte, R. L. Swain, Hellen Tappan, H. E. Tinnappel, H. S. Toney, E. P. Vance, R. W. Wagner, D. R. Whitney, R. B. Wildermuth, F. B. Wiley, C. O. Williamson, E. H. Wohler, Alberta Wolfe, G. F. Woodson, Jr.

The following officers were elected for the coming year: Chairman, E. P. Vance, Oberlin College; Secretary-Treasurer, Foster Brooks, Kent State University; Member of Executive Committee, R. L. Swain, Ohio State University; Program Committee, I. A. Barnett, University of Cincinnati (Chairman), L. C. Knight, Jr., Muskingum College, H. R. Mathias, Bowling Green State University.

The following papers were presented:

1. *Some current questions in administration of college mathematics*, by Professor R. H. Marquis, Ohio University.

The speaker made the following points: Textbooks written for high school students of elementary algebra and plane geometry are not suited to the more mature students required to take these subjects in college, hence there is a need for new texts for them. There is a great deal of

duplication in courses in elementary statistics given in various departments. The requirement of a course in the fundamentals given by the mathematics department would permit better courses to be built by the specialists in the other fields. The problems of securing a well-trained staff are still acute but becoming easier. The high schools will soon resume the trend toward the masters degree for the better jobs. The field of education is the easiest field for the schoolman to follow for this degree. A better approach to the needs of these men is needed on the part of mathematicians if they are to share in the advanced preparation of secondary teachers.

2. *Dynamics of variable mass systems*, by Professor L. F. Doty, University of Cincinnati (introduced by Professor I. A. Barnett).

It is known that the classical laws of motion of Newtonian mechanics require special interpretation when applied to a particle of varying mass. This concept is reviewed and extended to a system of particles of variable mass. By adopting a postulate concerning the continuous distribution of matter, vector equations of motion are derived for a finite body whose mass is a function of the time. As a special case, the scalar equations of plane motion are exhibited and compared with the usual rigid body equations.

3. *Some extensions of a geometrical construction problem*, by Mr. L. L. Ross, Ohio State University (introduced by the Secretary).

This paper presents several distinct solutions of the problem: Given four points, to construct a line through each such that they intersect to form a square. The number of solutions, and some special cases, are also discussed.

One of the methods used in solving the square problem is extended to solve these related problems: (1) Construct a quadrilateral similar to a given one so that each side contains one of four given points; (2) Construct lines tangent to each of four given circles so that the lines form a quadrilateral similar to a given one; (3) In a fixed quadrilateral, inscribe a quadrilateral similar to a given one.

By considering a triangle as a special case of a quadrilateral, one obtains solutions to problems such as that of constructing a triangle, congruent to a given one, having each side tangent to one of three given circles.

4. *Linear periodic functions*, by Professor F. W. Reed, Ohio University.

The linear periodic functions (sinl t and cosl t and others derived from these by iteration) are used for writing the equations of figures made up of polygons and their degenerate forms. Segmental curves of cursive type are analysed into periodic and secular parts. Examples are: field of hexagons, packed circles, family of all regular polygons, checker-board. The methods yield equations of surfaces having plane and curved facets, including the polyhedrons, the crystals, toroids, etc. Other examples are: packed spheres, stacked cubes, and honey-comb surfaces.

5. *A machine for solving simultaneous linear equations*, by Mr. Edmund Churchill and Mr. Herman Berman, Antioch College (introduced by Professor Max Astrachan).

A machine for the solution of simultaneous linear equations by the transformation of the mathematical problem into an electrical problem was demonstrated. Multiplications and additions of currents needed for solution is accomplished by simple resistances and potentiometers. Circuit diagrams together with directions for the construction of a model (cost about $35.00) for the solution of two equations (can be extended to more variables) are available from the authors.

6. *History of mathematics for secondary school teachers*, by Professor O. L. Dustheimer, University of Toledo.

After a short review of the general plan of a course for young teachers of high school mathe-

matics, several allied topics were discussed. The real history of elementary mathematics centers around fifty-seven mathematicians. The special topics mentioned were: old, new, and world calendars, square and cube roots by arithmetic, $\sqrt{2}$, $\sqrt{3}$, π, insurance and annuities, and determination of the date of Easter.

7. *Discussion of classroom notes.*

This discussion was led by Professor Wayne Dancer, University of Toledo, with contributions by R. L. Swain, I. A. Barnett, F. B. Wiley, H. S. Toney and Tibor Rado.

8. *Some questions in metric geometry*, By Professor E. J. Mickle, Ohio State University.

Professor Mickle's paper was a one-hour address delivered by invitation of the Program Committee. Let T and T^* be two continuous mappings from a unit square Q in a plane into euclidean three space E_3. Assume that for every pair of points in Q the images under T^* are never farther apart than the corresponding images under T. Then T and T^* are representations of surfaces whose Lebesgue areas $A(T)$ and $A(T^*)$ satisfy the inequality $A(T^*) \leqq A(T)$. The proof of this inequality utilizes the fact that a Lipschitzian transformation with constant k from a set in a euclidean space into a euclidean space can be extended to a Lipschitzian transformation with the same constant k defined on the whole space. Recent work in surface area utilizes a similar result on the extension of a Lipschitzian transformation in Banach spaces.

FOSTER BROOKS, *Secretary*

CALENDAR OF FUTURE MEETINGS

International Congress of Mathematicians, Cambridge, Massachusetts, August 30–September 6, 1950.

Thirty-fourth Annual Meeting, University of Florida, Gainesville, December, 1950.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN

ILLINOIS, Southern Illinois University, Carbondale, May 12–13, 1950.

INDIANA, Wabash College, Crawfordsville, April 29, 1950.

IOWA, State University of Iowa, Iowa City, April 21–22, 1950.

KANSAS, Spring, 1950.

KENTUCKY, University of Kentucky, Lexington, April 29, 1950.

LOUISIANA-MISSISSIPPI, Centenary College, Shreveport, Louisiana, February 24–25, 1950.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA.

METROPOLITAN NEW YORK, Spring, 1950.

MICHIGAN, March, 1950.

MINNESOTA, Macalester College, St. Paul, May 6, 1950.

MISSOURI, Spring, 1950.

NEBRASKA, Nebraska Wesleyan University, Lincoln, May 6, 1950.

NORTHERN CALIFORNIA, Berkeley, January 28, 1950.

OHIO, Denison University, Granville, April 22, 1950.

OKLAHOMA

PACIFIC NORTHWEST, University of Washington, Seattle, June, 1950.

PHILADELPHIA

ROCKY MOUNTAIN, University of Denver, April, 1950.

SOUTHEASTERN, University of Florida, Gainesville, April 7–8, 1950.

SOUTHERN CALIFORNIA, Immaculate Heart College, Hollywood, March 11, 1950.

SOUTHWESTERN, Spring, 1950.

TEXAS, Abilene, Spring, 1950.

UPPER NEW YORK, Syracuse University, Spring, 1950.

WISCONSIN, Marquette University, Milwaukee, May, 1950.

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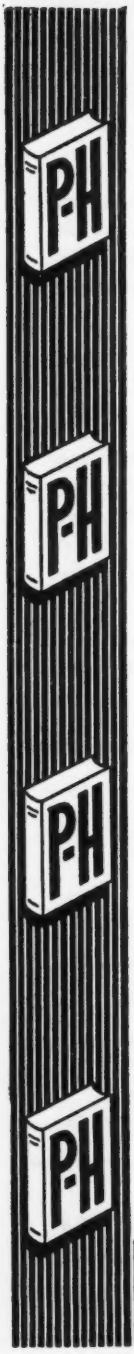

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